

# About dual one-dimensional oscillator and Coulomb-like theories.

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## Abstract

We present a mathematically rigorous quantum-mechanical treatment of a one-dimensional nonrelativistic quantum dual theories (with oscillator and Coulomb like potentials) and compare their spectra and the sets of eigenfunctions. We construct all self-adjoint Schrodinger operators for these theories and represent rigorous solutions of the corresponding spectral problems. Solving the first part of the problem, we use a method of specifying s.a. extensions by (asymptotic) s.a. boundary conditions. Solving spectral problems, we follow the Krein's method of guiding functionals. We show, that there is one to one correspondence between the spectral points of dual theories in the planes energy-coupling constants not only for discrete, but also for continuous spectra.

## 1 Introduction

It is well known [1], that if one introduces in a radial part of the  $D$  dimensional oscillator ( $D > 2$ )

$$\frac{d^2 R}{du^2} + \frac{D-1}{u} \frac{dR}{du} - \frac{L(L+D-2)}{u^2} R + \frac{2\mu}{\hbar^2} \left( E - \frac{\mu\omega^2 u^2}{2} \right) R = 0 \quad (1.1)$$

(here  $R$  is the radial part of the wave function for the  $D$  dimensional oscillator ( $D > 2$ ) and  $L = 0, 1, 2, \dots$  are the eigenvalues of the global angular momentum )  $r = u^2$  then equation (1.1) transforms into

$$\frac{d^2 R}{dr^2} + \frac{d-1}{r} \frac{dR}{dr} - \frac{l(l+d-2)}{r^2} R + \frac{2\mu}{\hbar^2} \left( \mathcal{E} + \frac{\alpha}{r} \right) R = 0 \quad (1.2)$$

where  $d = D/2 + 1$ ,  $l = L/2$ ,  $\mathcal{E} = -\frac{\mu\omega^2}{8}$ ,  $\alpha = E/4$ , which formally is identical to the radial equation for  $d$ -dimensional hydrogen atom.

Equations (1.1) and (1.2) are dual to each other and the duality transformation is  $r = u^2$ . For discrete spectrum of these equations it was proved, that to each state of equation (1.1) corresponds a state in (1.2), and visa versa [2, 3]. However the correspondence of the states in general (for discrete, as well as continuous spectra and for all values of the parameters of

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the theory) the problems was not considered. In this article we will consider the problem for the one-dimensional case, in which the Schrödinger equation for the oscillator is

$$\frac{d^2\Psi}{du^2} + \left( \frac{2\mu}{\hbar^2} E_O - \lambda u^2 \right) \Psi = 0, \quad \lambda = \frac{\mu^2 \omega^2}{\hbar^2} \quad (1.3)$$

which under duality transformation  $x = u|u|$  and setting

$$\Psi = x^{-1/4} \Phi \quad (1.4)$$

transforms into

$$\frac{d^2\Phi}{dx^2} + \left( \frac{2\mu}{\hbar^2} E_C - \frac{g}{|x|} + \frac{3}{16x^2} \right) \Phi = 0, \quad (1.5)$$

where  $E_C$  and  $g$  are some functions of parameters  $E_O$  and  $\lambda$ .

Eq.(1.5) includes a Coulomb-like potential and describes the so called  $1D$  anyon. Unlike the Eq.(1.3), which is defined for all values of the variable, eq.(1.5) is defined on the axis with punctured zero point. Taking into account, that duality transformation  $x = u|u|$  is also singular at the origin, we will consider the oscillator problem also with punctured zero point.

We will solve the quantum problem of these two equation and will show a complete correspondence of the states for all values of the parameters  $E_O$ ,  $\lambda$ ,  $E_C$ , and  $g$ . In section 2 we will consider the quantum problem for the oscillator, will find solutions of the equation for all values of the variable and parameters. In Section 3 we will consider the quantum problem for Coulomb-like system. The results will be compared in section 4, where we will show the one-to one correspondence of the spectra and proper functions of the Hamiltonians of both problems.

## 2 Quantum one-dimentional oscillator

We consider an equation

$$\partial_u^2 \psi(u) + (W - \lambda u^2) \psi(u) = 0, \quad (2.1)$$

where  $\hbar^2 W / 2\mu$  is complex energy,  $\hbar^2 \lambda / 2\mu$  is a coupling constant,

$$W = |W| e^{i\varphi_W}, \quad +0 \leq \varphi_W \leq \pi - 0, \quad \text{Im } W \geq 0,$$

It is convenient to write  $\lambda = \varkappa^4$ , where

$$\varkappa = \begin{cases} \lambda^{1/4}, & \varkappa^2 = \lambda^{1/2}, \quad \lambda \geq 0 \\ e^{-i\pi/4} |\lambda|^{1/4}, & \varkappa^2 = e^{-i\pi/2} |\lambda|^{1/2}, \quad \lambda < 0 \end{cases}.$$

### 2.1 Solutions on the semiaxis $u > 0$

To find the solutions on the semiaxis  $u > 0$ , we will introduce a new variable  $\rho = (\varkappa u)^2$ ,  $\partial_u = 2\varkappa\sqrt{\rho}\partial_\rho$ ,  $\partial_u^2 = 4[\rho\partial_\rho^2 + (1/2)\partial_\rho]$ , and new function  $\phi(\rho) = e^{\rho/2}\psi(u)$ . Then we obtain

$$\rho\partial_\rho^2 \phi(\rho) + (1/2 - \rho)\partial_\rho \phi(\rho) - (1/4 - w)\phi(\rho) = 0, \quad w \equiv w_O = W/4\varkappa^2. \quad (2.2)$$

Eq. (2.2) is the equation for confluent hypergeometric functions with solutions  $\Phi(\alpha, \gamma; \rho)$ ,  $\Psi(\alpha, \gamma; \rho)$ , in the terms of which we can express solutions of eq. (2.1). We will use the following solutions:

$$\begin{aligned} O_{+1}(u; W) &= \frac{1}{\varkappa} \sqrt{\rho} e^{-\rho/2} \Phi(\alpha + 1/2, 3/2; \rho), \\ O_{+2}(u; W) &= e^{-\rho/2} \Phi(\alpha, 1/2; \rho), \\ O_{+3}(u; W) &= \pi^{-1/2} \Gamma(\alpha + 1/2) e^{-\rho/2} \Psi(\alpha, 1/2; \rho) = \\ &= O_{+2}(u; W) - \frac{2\varkappa \Gamma(\alpha + 1/2)}{\Gamma(\alpha)} O_{+1}(u; W), \quad \alpha + 1/2 \neq -n, n \in \mathbb{Z}_+, \\ \alpha &\equiv \alpha_O = 1/4 - w. \end{aligned}$$

In this section, we will omit the subscript “ $O$ ” meaning, for example,  $\alpha \equiv \alpha_O$ ,  $w \equiv w_O$ , and so on.

### 2.1.1 Asymptotics

For  $u \rightarrow 0$  we get

$$\begin{aligned} O_{+1}(u; W) &= u + O(u^3), \quad O_{+2}(u; W) = 1 + O(u^2), \\ O_{+3}(u; W) &= 1 - \frac{2\varkappa \Gamma(\alpha + 1/2)}{\Gamma(\alpha)} u + O(u^2), \quad \alpha + 1/2 \neq -n, \quad n \in \mathbb{Z}_+. \end{aligned}$$

The asymptotics for  $u \rightarrow \infty$  and different values of the parameters we get  $\lambda > 0$ ,  $\text{Im } w > 0$  or  $w = 0$

$$\begin{aligned} O_{+1}(u; W) &= \frac{\sqrt{\pi}}{2\varkappa \Gamma(\alpha + 1/2)} \rho^{-1/4-w} e^{\rho/2} (1 + O(u^{-2})), \\ O_{+2}(u; W) &= \frac{\sqrt{\pi}}{\Gamma(\alpha)} \rho^{-1/4-w} e^{\rho/2} (1 + O(u^{-2})), \\ O_{+3}(u; W) &= \frac{\Gamma(\alpha + 1/2)}{\sqrt{\pi}} \rho^{-1/4+w} e^{-\rho/2} (1 + O(u^{-2})), \end{aligned}$$

$\lambda < 0$ ,  $\text{Im } W > 0$  or  $W = 0$

$$\begin{aligned} O_{+1}(u; W) &= O(u^{-1/2+\text{Im } W/2\sqrt{|\lambda|}}), \quad O_{+2}(u; W) = O(u^{-1/2+\text{Im } W/2\sqrt{|\lambda|}}), \\ O_{+3}(u; W) &= O(u^{-1/2-\text{Im } W/2\sqrt{|\lambda|}}). \end{aligned}$$

The asymptotics for  $\lambda = 0$  can be obtained as a limit  $\lambda \rightarrow 0$  of corresponding formulae or from explicit expressions for solutions as  $\lambda = 0$ ,

$$\begin{aligned} \{O_{+1}(u; W)\}_{\lambda=0} &= u \Phi(-W/4\varkappa^2, 3/2; \varkappa^2 u^2) = \frac{1}{\sqrt{W}} \sin(\sqrt{W}u), \\ \{O_{+2}(u; W)\}_{\lambda=0} &= \Phi(-W/4\varkappa^2, 1/2; \varkappa^2 u^2) = \cos(\sqrt{W}u), \\ \{O_{+3}(u; W)\}_{\lambda=0} &= \cos(\sqrt{W}u) - \frac{2\varkappa \Gamma(\alpha + 1/2)}{\Gamma(\alpha)} \Big|_{\varkappa \rightarrow 0} \frac{1}{\sqrt{W}} \sin(\sqrt{W}u) = e^{i\sqrt{W}u} \end{aligned}$$

(where we used a relation  $2\kappa\Gamma(\alpha + 1/2)/\Gamma(\alpha)|_{\kappa \rightarrow 0} = \sqrt{-W} = -i\sqrt{W}$ ), which are in agreement with direct solution of eq. (2.1) for  $\lambda = 0$ .

Note, that all solutions of eq. (2.1) are square-integrable at the origin and only solution,  $O_{+3}(u; W)$ , is square-integrable at infinity for  $\text{Im } W > 0$ , i. e.,  $O_{+3}(u; W) \in L^2(\mathbb{R}_+)$  for  $\text{Im } W > 0$ .

The functions  $O_{+1}$  and  $O_{+2}$  are entire functions of  $W$  ( for fixed rest parameters and  $u$ ). They are real for real  $W$  and nonnegative  $\lambda$ . If  $\lambda$  is negative, then  $\kappa^2$  is pure imaginary and changes sign under complex conjugation. But the functions  $O_{+1}$  and  $O_{+2}$  are even functions of  $\kappa^2$ , that follows from the relation 9.212.1 of [4]

$$\begin{aligned} e^{-\varrho/2}\Phi(\alpha_O, 1/2; z) &= e^{-\kappa^2 u^2/2}\Phi(1/4 - W/4\kappa^2, 1/2; \kappa^2 u^2) = \\ &= e^{\rho/2}\Phi(1/2 - \alpha, 1/2; -\rho) = e^{\kappa^2 u^2/2}\Phi(1/4 + W/4\kappa^2, 1/2; -\kappa^2 u^2), \\ e^{-\rho/2}\Phi(\alpha + 1/2, 3/2; \rho) &= e^{-\kappa^2 u^2/2}\Phi(3/4 - W/4\kappa^2, 3/2; \kappa^2 u^2) = \\ &= e^{\rho/2}\Phi(3/2 - \alpha - 1/2, 3/2; -\rho) = e^{\kappa^2 u^2/2}\Phi(3/4 + W/4\kappa^2, 3/2; -\kappa^2 u^2). \end{aligned}$$

Thus, we find that the functions  $O_{+1}$  and  $O_{+2}$  are real-entire in  $W$  for all  $\lambda$ .

Finally, using the asymptotics for  $u \rightarrow 0$  we find the Wronskians of the solutions

$$\begin{aligned} \text{Wr}(O_{+1}, O_{+2}) &= \text{Wr}(O_{+1}, O_{+3}) = -1, \\ \text{Wr}(O_{+2}, O_{+3}) &= -2\kappa \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha)}. \end{aligned}$$

### 2.1.2 Solution on the semiaxis $u < 0$

For  $u < 0$ , we will use the solutions  $O_{-k}(u; W)$ ,

$$O_{-k}(u; W) = O_{+k}(|u|; W), \quad k = 1, 2, 3, \quad u < 0.$$

## 2.2 Symmetrical operator $\hat{H}_O$

For given a differential operation  $\check{H}_O$  ( $\check{H}$  in what follows in this section),

$$\check{H} = -\partial_u^2 + \lambda u^2, \tag{2.3}$$

we determine the following symmetrical operator  $\hat{H}_O \equiv \hat{H}$ ,

$$\hat{H} : \begin{cases} D_H = \mathcal{D}(\mathbb{R} \setminus \{0\}), \\ \hat{H}\psi(u) = \check{H}\psi(u), \quad \forall \psi \in D_H, \end{cases}$$

where  $\mathcal{D}(\Delta)$  is a space of smooth functions with a compact support (i.e. which are equal to zero in some neighbourhoods of the endpoints of the interval  $\Delta$ ).

## 2.3 Adjoint operator $\hat{H}_O^+ = \hat{H}_O^*$

The adjoint operator  $\hat{H}_O^+$  is

$$\hat{H}_O^+ \equiv \hat{H}^+ : \begin{cases} D_{H^+} = \{\psi_*, \psi_*' \text{ are a.c. on } \mathbb{R} \setminus \{0\}, \psi_*, \hat{H}^+\psi_* \in L^2(\mathbb{R})\} \\ \hat{H}^+\psi_*(u) = \check{H}\psi_*(u), \quad u \in \mathbb{R} \setminus \{0\}, \quad \forall \psi_* \in D_{H^+} \end{cases}.$$

where a.c. means absolutely continuous.

### 2.3.1 Asymptotics of $\psi_* \in D_{H^+}$

I)  $|u| \rightarrow \infty$

Because  $V(u) = \lambda u^2 > -(|\lambda|+1)u^2$ , we have:  $[\psi_*, \chi_*](u) \rightarrow 0$  as  $u \rightarrow \pm\infty, \forall \psi_*, \chi_* \in D_{H^+}$  [5].

Here  $[\chi_*, \psi_*](u) = \overline{\chi'_*(u)}\psi_*(u) - \overline{\chi_*(u)}\psi'_*(u)$ .

II)  $u \rightarrow +0$

Because  $\check{H}\psi_* \in L^2(\mathbb{R})$ , we have

$$\check{H}\psi_*(u) = (-\partial_u^2 + \lambda u^2)\psi_*(u) = \eta(u), \quad \eta \in L^2(\mathbb{R}).$$

General solution of this equation can be represented in the form

$$\begin{aligned} \psi_*(u) &= a_{+1}O_{+1}(u; 0) + a_{+2}O_{+2}(u; 0) + I(u), \\ \psi'_*(u) &= a_{+1}O'_{+1}(u; 0) + a_{+2}O'_{+2}(u; 0) + I'(u), \end{aligned}$$

where

$$\begin{aligned} I(u) &= O_{+2}(u; 0) \int_0^u O_{+1}(v; 0)\eta(v)dv - O_{+1}(u; 0) \int_0^u O_{+2}(v; 0)\eta(v)dv, \\ I'(u) &= O'_{+2}(u; 0) \int_0^u O_{+1}(v; 0)\eta(v)dv - O'_{+1}(u; 0) \int_0^u O_{+2}(v; 0)\eta(v)dv. \end{aligned}$$

We obtain with the help of the Cauchy-Bunyakovskii inequality:

$$I(u) = O(u^{3/2}), \quad I'(u) = O(u^{1/2}), \quad u \rightarrow +0,$$

such that we find

$$\begin{aligned} \psi_*(u) &= a_{+1}u + a_{+2} + O(u^{3/2}), \quad \psi'_*(u) = a_{+1} + O(u^{1/2}), \quad u \rightarrow +0, \\ a_{+2} &= \psi_*(+0), \quad a_{+1} = \psi'_*(+0). \end{aligned}$$

III)  $u \rightarrow -0$  Analogously, we obtain for  $u \rightarrow -0$ :

$$\begin{aligned} \psi_*(u) &= -a_{-1}u + a_{-2} + O(|u|^{3/2}), \quad \psi'_*(u) = -a_{-1} + O(|u|^{1/2}), \quad u \rightarrow -0, \\ a_{-2} &= \psi_*(-0), \quad a_{-1} = -\psi'_*(-0). \end{aligned}$$

## 2.4 Sesquilinear form

Sesquilinear form of adjoint operator  $\hat{H}^+$ ,  $\omega_{H^+}(\psi_*, \chi_*)$  is defined as

$$\begin{aligned} \omega_{H^+}(\chi_*, \psi_*) &= \omega_{+H^+}(\chi_*, \psi_*) + \omega_{-H^+}(\chi_*, \psi_*), \\ \omega_{+H^+}(\chi_*, \psi_*) &= \int_0^\infty \left[ \overline{\chi_*(u)}\check{H}\psi_*(u) - \overline{\check{H}\chi_*(u)}\psi_*(u) \right] du = \\ &= -[\chi_*, \psi_*](u)|_{u \rightarrow +0} = \overline{a_{\chi_*+2}}a_{\psi_*+1} - \overline{a_{\chi_*+1}}a_{\psi_*+2}, \\ \omega_{-H^+}(\chi_*, \psi_*) &= \int_{-\infty}^0 \left[ \overline{\chi_*(u)}\check{H}\psi_*(u) - \overline{\check{H}\chi_*(u)}\psi_*(u) \right] du = \\ &= [\chi_*, \psi_*](u)|_{u \rightarrow -0} = \overline{a_{\chi_*-2}}a_{\psi_*-1} - \overline{a_{\chi_*-1}}a_{\psi_*-2}, \\ [\chi_*, \psi_*](u) &= \overline{\chi'_*(u)}\psi_*(u) - \overline{\chi_*(u)}\psi'_*(u). \end{aligned}$$

Thus we have

$$\begin{aligned}\omega_{H_O^+}(\chi_*, \psi_*) &= \overline{\mathbf{a}_{\chi_* 2}} \mathbf{a}_{\psi_* 1} - \overline{\mathbf{a}_{\chi_* 1}} \mathbf{a}_{\psi_* 2} = \frac{i}{2\kappa_0} (\overline{\mathbf{b}_{\chi_*}} \mathbf{b}_{\psi_*} - \overline{\mathbf{d}_{\chi_*}} \mathbf{d}_{\psi_*}), \\ \mathbf{a}_1 &= \begin{pmatrix} a_{+1} \\ a_{-1} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} a_{+2} \\ a_{-2} \end{pmatrix}, \\ \mathbf{b} &= \begin{pmatrix} b_+ \\ b_- \end{pmatrix} = \mathbf{a}_1 + i\kappa_0 \mathbf{a}_2, \quad \mathbf{d} = \begin{pmatrix} d_+ \\ d_- \end{pmatrix} = \mathbf{a}_1 - i\kappa_0 \mathbf{a}_2,\end{aligned}$$

where  $\kappa_0$  is arbitrary, but fixed parameter of dimensionality of inverse length introduced by dimensional reasons.

## 2.5 Self-adjoint hamiltonians

Because all self-adjoint (s.a.) hamiltonians,  $\hat{H}_{O_\epsilon}$  ( $\equiv \hat{H}_\epsilon$ ), act on their domains as  $\check{H}$ , we should specify definition domains only. The definition domain  $D_{H_\epsilon}$  of s.a. operator  $\hat{H}_\epsilon$  is determined by condition

$$\omega_{H^+}(\chi, \psi) = 0, \quad \forall \chi, \psi \in D_{H_\epsilon},$$

from which it follows

$$\mathbf{d}_\psi = U \mathbf{b}_\psi, \quad \forall \psi \in D_{H_\epsilon}, \quad (2.4)$$

where  $U$  is an arbitrary, but fixed for given extension, unitary  $(2 \times 2)$ -matrix,  $U^+ U = 1$ . Thus, any s.a. hamiltonian is determined by assignment of unitary matrix  $U$  (we will denote the corresponding s.a. hamiltonian by  $\hat{H}_{OU} \equiv \hat{H}_U$ ),

$$\hat{H}_U^+ : \begin{cases} D_{H_U} \equiv D_U = \{\psi : \psi \in D_{\check{H}}^*, \mathbf{d}_\psi = U \mathbf{b}_\psi\} \\ \hat{H}_U \psi(u) = \check{H} \psi(u), \quad u \in \mathbb{R} \setminus \{0\}, \quad \forall \psi \in D_U \end{cases}.$$

Thus, there exists a  $U(2)$ -family of s.a. extensions of the initial symmetric operator  $\hat{H}$ .

## 2.6 Parity conserving extensions

We will further restrict ourselves to the s.a. extensions conserving parity,  $[\hat{P}, \hat{H}_U] = 0$ , where  $\hat{P}$  is the parity operator that acts on functions  $\psi(x)$  in  $L^2(\mathbb{R})$  as

$$\hat{P} \psi(u) = \psi(-u). \quad (2.5)$$

The Hilbert space  $L^2(\mathbb{R})$  can be decomposed in the direct orthogonal sum of a subspace  $L_s^2(\mathbb{R})$  symmetric functions and a subspace  $L_a^2(\mathbb{R})$  of antisymmetric functions, such that  $L^2(\mathbb{R}) = L_s^2(\mathbb{R}) \oplus L_a^2(\mathbb{R})$ ,

$$\begin{aligned}\psi &\in L^2(\mathbb{R}), \quad \psi = \psi_s + \psi_a, \quad \psi_s \in L_s^2(\mathbb{R}), \quad \psi_a \in L_a^2(\mathbb{R}), \\ \hat{P} \psi_s &= \psi_s, \quad \hat{P} \psi_a = -\psi_a.\end{aligned}$$

One can easily see that operators  $\hat{H}_O$  and  $\hat{H}_O^+$  commute with  $\hat{P}$ ,

$$[\hat{P}, \hat{H}] = [\hat{P}, \hat{H}^+] = 0.$$

This means that the operators  $\hat{H}$  and  $\hat{H}^+$  can be represented in the form of direct sum of their parts in the corresponding subdomains of symmetric and antisymmetric functions:

$$\begin{aligned}\hat{H} &= \hat{H}_s \oplus \hat{H}_a, \quad D_H = D_{Hs} \oplus D_{Ha}, \quad D_{Hs,a} = \mathcal{D}_{s,a}(\mathbb{R} \setminus \{0\}), \\ \hat{H}\psi &= \hat{H}_s\psi_s + \hat{H}_a\psi_a, \quad \psi = \psi_s + \psi_a, \quad \psi \in D_H, \quad \psi_{s,a} \in D_{Hs,a},\end{aligned}$$

where  $\mathcal{D}_{s,a}(\mathbb{R} \setminus \{0\})$  are subspaces of symmetric (antisymmetric) functions in  $\mathcal{D}(\mathbb{R} \setminus \{0\})$ . Similar decompositions hold true for the adjoint operator  $\hat{H}^+$ ,

$$\hat{H}^+ = \hat{H}_s^+ \oplus \hat{H}_a^+, \quad D_{H^+}(\mathbb{R} \setminus \{0\}) = D_{H^+}(\mathbb{R} \setminus \{0\})_s \oplus D_{H^+}(\mathbb{R} \setminus \{0\})_a,$$

where  $D_{H^+}(\mathbb{R} \setminus \{0\})_{s,a}$  are subspaces of symmetric (antisymmetric) functions in  $D_{H^+}(\mathbb{R} \setminus \{0\})$ :

$$D_{H^+}(\mathbb{R} \setminus \{0\})_{s,a} = \left\{ \psi_{*s,a} : \psi_{*s,a}, \psi'_{*s,a} \text{ are a.c. on } \mathbb{R} \setminus \{0\}, \psi_{*s,a}, \hat{H}_{s,a}^+ \psi_{*s,a} \in L_{s,a}^2(\mathbb{R}) \right\}$$

Because the operator  $\hat{P}$  is bounded,  $\|\hat{P}\| = 1$ , and  $\hat{P}^2 = 1$ , the assertion that  $\hat{P}$  commutes with  $\hat{H}_U$  means

$$[\hat{P}, \hat{H}_U] = 0 \implies \hat{H}_U = \hat{H}_{sU} \oplus \hat{H}_{aU},$$

where operators  $\hat{H}_{s,aU}$  are s.a. extensions of the operators  $\hat{H}_{s,a}$ . In turn, if  $\hat{H}_{s,aU}$  are s.a. extensions of  $\hat{H}_{s,a}$  in  $L_{s,a}^2(\mathbb{R})$ , then the operator  $\hat{H}_U = \hat{H}_{sU} \oplus \hat{H}_{aU}$  is a s.a. extension of  $\hat{H}$  in  $L^2(\mathbb{R})$  which commutes with  $\hat{P}$ . Thus, it is enough to describe all s.a. extensions of operators  $\hat{H}_{s,a}$  in the subspaces  $L_{s,a}^2(\mathbb{R})$  to find all commuting with  $\hat{P}$  s.a. extensions  $\hat{H}_U$  of the operator  $\hat{H}$ .

First, we find the general form of matrix  $U = U_P$  conserving (commuting with) the parity  $\hat{P}$ .

The condition:  $U_P$  commutes with  $\hat{P}$ , means that rel. (2.4) is valid for the functions  $\psi_{s,a} \in L_{s,a}^2(\mathbb{R})$ . The functions  $\psi_{s,a}$  have the properties

$$a_{s,a-2} = \pm a_{s,a+2}, \quad a_{s,a-1} = \pm a_{s,a+1}, \quad (2.6)$$

such that doublets  $\mathbf{d}_{\psi_{s,a}}$  and  $\mathbf{b}_{\psi_{s,a}}$  have the form,

$$\begin{aligned}\mathbf{b}_{\psi_{s,a}} &= \sqrt{2}[a_{s,a+1} + i\kappa_0 a_{s,a+2}] \mathbf{n}_{s,a}, \quad \mathbf{d}_{\psi_{s,a}} = \sqrt{2}[a_{s,a+1} - i\kappa_0 a_{s,a+2}] \mathbf{n}_{s,a}, \\ \mathbf{n}_s &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{n}_a = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.\end{aligned}$$

The condition (2.4) gives for such doublets

$$\begin{aligned}U_P \mathbf{n}_{s,a} &= \lambda_{s,a} \mathbf{n}_{s,a}, \\ \lambda_{s,a} &= \frac{a_{s,a+1} - i\kappa_0 a_{s,a+2}}{a_{s,a+1} + i\kappa_0 a_{s,a+2}} = \frac{\psi'_{s,a}(+0) - i\kappa_0 \psi_{s,a}(+0)}{\psi'_{s,a}(+0) + i\kappa_0 \psi_{s,a}(+0)} = e^{i\varphi_{s,a}}, \quad 0 \leq \varphi_{s,a} \leq 2\pi,\end{aligned} \quad (2.7)$$

i. e., orthonormalized vectors  $\mathbf{n}_{s,a}$  must be eigenvectors of matrix  $U$ . General form of matrices  $U = U_P$  satisfying condition (2.7) is

$$U_P = \lambda_s \mathbf{n}_s \otimes \mathbf{n}_s + \lambda_a \mathbf{n}_a \otimes \mathbf{n}_a. \quad (2.8)$$

The inverse statement is true as well. Namely, if matrix  $U$  has the form (2.8) then the subspaces  $L_{s,a}^2(\mathbb{R})$  reduce the corresponding s.a. hamiltonian  $\hat{H}_{U_P}$ , i. e., the hamiltonian  $\hat{H}_{U_P}$  commutes with parity operator  $\hat{P}$ .

In the terms of the asymptotical boundary (a.b.) conditions, such a form of the matrix  $U_P$  means the following:

$$a_{s,a+1} \cos \zeta_{s,a} = \kappa_0 a_{s,a+2} \sin \zeta_{s,a}, \quad |\zeta_{s,a}| \leq \pi/2, \quad \zeta_{s,a} = -\pi/2 \sim \zeta_{s,a} = \pi/2, \quad (2.9)$$

or

$$\psi_{s,a}(u) = \begin{cases} a(\kappa_0 u \sin \zeta_{s,a} + \cos \zeta_{s,a}) + O(u^{3/2}), & u > 0 \\ \pm a(\kappa_0 |u| \sin \zeta_{s,a} + \cos \zeta_{s,a}) + O(u^{3/2}), & u < 0 \end{cases}, \quad u \rightarrow 0, \quad (2.10)$$

where  $\zeta_{s,a} = \varphi_{s,a}/2 - \pi/2$ . The inverse statement is true as well. Namely, if matrix  $U$  gives the boundary condition of the form (2.10) (or (2.9)) then that matrix  $U$  has the form (2.8) with  $\varphi_{s,a} = 2\zeta_{s,a} + \pi$ . In what follows, we change the notation of s.a. operator  $\hat{H}_{U_P}$  for  $\hat{H}_{\zeta_{s,a}}$ .

## 2.7 Extensions on semiaxis $\mathbb{R}_+$

To extend the adjoint operator on semiaxis  $\mathbb{R}_+$  define for the differential operation  $\check{h}_O (\equiv \check{h})$

$$\check{h} = \check{H} = -\partial_u^2 + \lambda u^2,$$

a symmetrical operator  $\hat{h}_O (\equiv \hat{h})$

$$\hat{h} : \begin{cases} D_h = \mathcal{D}(\mathbb{R}_+) \\ \hat{h}\psi(u) = \check{h}\psi(u), \quad \forall \psi \in D_h, \end{cases}$$

and the adjoint operator  $\hat{h}_O^+$

$$\hat{h}_O^+ \equiv \hat{h}^+ : \begin{cases} D_{h^+} = \{\psi_*, \psi'_* \text{ are a.c. on } \mathbb{R}_+, \psi_*, \check{H}_O \psi_* \in L^2(\mathbb{R}_+)\} \\ \hat{h}^+ \psi_*(u) = \check{h}\psi_*(u), \quad \forall \psi_* \in D_{h^+} \end{cases}.$$

Literally repeating the considerations of subsec.2.3.1 we obtain the asymptotics:

I)  $u \rightarrow \infty$

$$[\psi_*, \chi_*](u) \rightarrow 0, \quad \forall \psi_*, \chi_* \in D_{h^+}.$$

II)  $u \rightarrow 0$

$$\begin{aligned} \psi_*(u) &= a_1 u + a_2 + O(u^{3/2}), \quad \psi'_*(u) = a_1 + O(u^{1/2}), \\ a_2 &= \psi_*(0), \quad a_1 = \psi'_*(0). \end{aligned}$$

For the sesquilinear form  $\omega_{h^+}(\psi_*, \chi_*)$  we get

$$\begin{aligned} \omega_{h^+}(\chi_*, \psi_*) &= \int_0^\infty \left[ \overline{\chi_*(u)} \check{h}\psi_*(u) - \overline{\check{h}\chi_*(u)} \psi_*(u) \right] du = \\ &= - [\chi_*, \psi_*](u)|_{u \rightarrow 0} = \overline{a_{\chi_* 2}} a_{\psi_* 1} - \overline{a_{\chi_* 1}} a_{\psi_* 2} = \\ &= \frac{i}{2\kappa_0} (\overline{b_{\chi_*}} b_{\psi_*} - \overline{d_{\chi_*}} d_{\psi_*}), \quad b = a_1 + i\kappa_0 a_2, \quad d = a_1 - i\kappa_0 a_2. \end{aligned}$$



### 2.7.1 Self-adjoint hamiltonians

Because all s.a. hamiltonians,  $\hat{h}_{O\epsilon} (\equiv \hat{h}_\epsilon)$ , act on its domains as  $\check{h}$ , we should specify definition domains only. The definition domain  $D_{h_\epsilon}$  of s.a. operator  $\hat{h}_\epsilon$  is determined by condition

$$\omega_{h^+}(\chi, \psi) = 0, \quad \forall \chi, \psi \in D_{h_\epsilon},$$

from which it follows

$$d_\psi = e^{i\varphi} b_\psi, \quad \forall \psi \in D_{h_\epsilon}, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \sim 2\pi,$$

or, equivalent

$$a_1 \cos \zeta = \kappa_0 a_2 \sin \zeta, \quad \zeta \in S(-\pi/2, \pi/2), \quad \zeta = \varphi/2 - \pi/2.$$

Thus, any s.a. hamiltonian is determined by assignment of unitary matrix  $U(1) = e^{i\varphi}$  (we will denote the corresponding s.a. hamiltonian by  $\hat{h}_{O\zeta}$ ),

$$\hat{h}_{O\zeta} \equiv \hat{h}_\zeta : \begin{cases} D_{h_\zeta} \equiv D_\zeta = \{\psi : \psi \in D_{\check{h}}^*, \quad a_1 \cos \zeta = \kappa_0 a_2 \sin \zeta\} \\ \hat{h}_\zeta \psi(u) = \check{h} \psi(u), \quad \forall \psi \in D_\zeta \end{cases}.$$

Equivalently, the boundary condition for  $\psi \in D_\zeta$  can be represented in the form

$$\psi(u) = a(\kappa_0 u \sin \zeta + \cos \zeta) + O(u^{3/2}), \quad u \rightarrow 0. \quad (2.11)$$

Thus, there exists a  $U(1)$ -family of s.a. extensions  $\hat{h}_\zeta$  of the initial symmetric operator  $\hat{h}$ .

## 2.8 Self-adjoint extensions of $\hat{H}_s$

The Hilbert space  $L_s^2(\mathbb{R})$  is the space of all symmetric functions that are square integrable on  $\mathbb{R}$ . These functions obey the relations

$$\psi(+0) = \psi(-0) = \psi(0), \quad \psi'(+0) = -\psi'(-0), \quad \forall \psi \in L_s^2(\mathbb{R}),$$

(see (2.6)) which implies

$$(\chi, \psi) = 2(\chi, \psi)_+, \quad \omega_{H^+}(\chi, \psi) = 2\omega_{H^+}(\chi, \psi)_+ = 2\omega_{h^+}(\chi, \psi), \quad (2.12)$$

where

$$(\chi, \psi)_+ = \int_0^\infty \overline{\chi(u)} \psi(u) du, \quad (2.13)$$

and  $\omega_{H^+}(\chi, \psi)_+ = \omega_{h^+}(\chi, \psi)$  is the sesquilinear form with respect to the scalar product (2.13).

Let us consider the isometry  $T: \psi \in \mathbb{R} \xrightarrow{T} \sqrt{2}\psi, \psi \in \mathbb{R}_+$ . Then

$$D_{H_s} \xrightarrow{T} D_h = \mathcal{D}(\mathbb{R}_+), \quad D_{H_s^+} = D_{\check{H}}^*(\mathbb{R} \setminus \{0\}) \xrightarrow{T} D_{h^+}. \quad (2.14)$$

It follows from eqs. (2.12) and (2.14) that there is one-to-one correspondence (the isometry  $T$ ) between s.a. extensions  $\hat{H}_{\zeta_s}$  of the symmetric operator  $\hat{H}_s$  in  $L_s^2(\mathbb{R})$  and s.a. extensions  $\hat{h}_\zeta$  of the symmetric operator  $\hat{h}$  in  $L^2(\mathbb{R}_+)$ :  $\hat{H}_{\zeta_s} \xleftrightarrow{T} \hat{h}_\zeta, \zeta_s = \zeta$ . Thus, the spectral analysis of s.a. operator  $\hat{H}_{\zeta_s}$  in  $L_s^2(\mathbb{R})$  is reduced to the spectral analysis of s.a. operator  $\hat{h}_\zeta, \zeta_s = \zeta$ , in  $L^2(\mathbb{R}_+)$ . Below, we represent this analysis.

### 2.8.1 Guiding functional

To perform the analysis we have to define the guiding functional  $\Phi_{O\zeta}(\xi; W) \equiv \Phi_\zeta(\xi; W)$  [5, 6]

$$\begin{aligned}\Phi_\zeta(\xi; W) &= \int_0^\infty U_\zeta(u; W) \xi(u) du, \quad \xi \in \mathbb{D}_\zeta = D_r(\mathbb{R}_+) \cap D_{h_\zeta}, \\ U_\zeta(u; W) &\equiv U_{O\zeta}(u; W) = \kappa_0 O_{+1}(u; W) \sin \zeta + O_{+2}(u; W) \cos \zeta, \\ D_r(a, b) &= \{\psi(u) : \text{supp} \psi \subseteq [a, \beta_\psi], \beta_\psi < b.\end{aligned}$$

Note that  $U_\zeta(u; W)$  is real-entire solution of eq. (2.1) and satisfies the boundary conditions (2.11).

The guiding functional  $\Phi_\zeta(\xi; W)$  satisfies the properties 1)- 3) of [5, 6]. We will call the guiding functional  $\Phi_\zeta(\xi; W)$  with those properties "simple". It follows [5, 6] that the spectrum of  $\hat{h}_\zeta$  is simple.

### 2.8.2 Green function $G_{O\zeta}(u, v; W)$ , spectral function $\sigma_{O\zeta}(E)$

The Green function  $G_{O\zeta}(u, v; W) \equiv G_\zeta(u, v; W)$  is the kernel of the integral representation

$$\psi(u) = \int_0^\infty G_\zeta(u, v; W) \eta(v) dv, \quad \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_\zeta^+ - W)\psi(u) = \eta(u), \quad \text{Im } W > 0, \quad (2.15)$$

for  $\psi \in D_\zeta$ , that is,  $\psi \in L^2(\mathbb{R}_+)$  and  $\psi$  satisfies the boundary conditions (2.11). We find

$$G_\zeta(u, v; W) = \Omega_\zeta(W) U_\zeta(u; W) U_\zeta(v; W) - \frac{1}{\kappa_0} \begin{cases} \tilde{U}_\zeta(u; W) U_\zeta(v; W), & u > v \\ U_\zeta(u; W) \tilde{U}_\zeta(v; W), & u < v \end{cases}, \quad (2.16)$$

$$\Omega_\zeta(W) \equiv \Omega_{O\zeta}(W) = \frac{\tilde{\omega}_\zeta(W)}{\kappa_0 \omega_\zeta(W)}, \quad \omega_{O\zeta}(W) \equiv \omega_\zeta(W) = \sin \zeta + \tilde{\gamma}(W) \cos \zeta,$$

$$\tilde{\omega}_\zeta(W) \equiv \tilde{\omega}_{O\zeta}(W) = \cos \zeta - \tilde{\gamma}(W) \sin \zeta, \quad \tilde{\gamma}(W) \equiv \tilde{\gamma}_O(W) = \frac{2\kappa}{\kappa_0} \gamma(\alpha),$$

$$\gamma(\alpha) \equiv \gamma_O(\alpha_O) = \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha)},$$

where we used relations

$$\begin{aligned}\kappa_0 O_{+1}(u; W) &= U_\zeta(u; W) \sin \zeta + \tilde{U}_\zeta(u; W) \cos \zeta, \\ O_{+2}(u; W) &= U_\zeta(u; W) \cos \zeta - \tilde{U}_\zeta(u; W) \sin \zeta, \\ O_{+3}(u; W) &= U_\zeta(u; W) \tilde{\omega}_\zeta(W) - \tilde{U}_\zeta(u; W) \omega_\zeta(W).\end{aligned}$$

Note that  $\tilde{U}_\zeta(u; W)$  is real-entire solution of eq. (2.1) such that the last term in the r.h.s. of eq. (2.16) is real for  $W = E_O \equiv E$  ( $\text{Im } W = 0$ ). From the relation [5, 6]

$$U_\zeta^2(u_0; E) \sigma'_\zeta(E) = \frac{1}{\pi} \text{Im } G_\zeta(u_0 - 0, u_0 + 0; E + i0),$$

where  $f(E + i0) \equiv \lim_{\varepsilon \rightarrow +0} f(E + i\varepsilon)$ ,  $\forall f(W)$ , we find

$$\sigma_{O\zeta}(E) \equiv \sigma'_\zeta(E) = \frac{1}{\pi} \text{Im } \Omega_\zeta(E + i0).$$

Now we proceed to defining the spectrum of the theory.

### 2.8.3 Spectrum, $\lambda > 0$

We have  $\varkappa = \lambda^{1/4} > 0$ ,  $W = E + i\varepsilon$ ,  $w(E) = w|_{W=E} = E/4\sqrt{\lambda}$

a)  $\zeta = \pi/2$

$$\Omega_{\pi/2}(W) = -\frac{\tilde{\gamma}(E + i\varepsilon)}{\kappa_0} = -\frac{2\lambda^{1/4}\Gamma(3/4 - (E + i\varepsilon)/4\sqrt{\lambda})}{\kappa_0^2\Gamma(1/4 - (E + i\varepsilon)/4\sqrt{\lambda})}$$

The function  $\Omega_{\pi/2}(E)$  is real for  $E$  where  $|\Omega_{\pi/2}(E)| < \infty$ . Therefore,  $\text{Im } \Omega_{\pi/2}(E + i0)$  can be not equal to zero only in the points  $\Omega_{\pi/2}(E) = \pm\infty$ , i.e., in the points  $\alpha + 1/2 = -n$ ,  $w(\vartheta_{On}) = w_{\pi/2|n} = n + 3/4$ ,  $n \in \mathbb{Z}_+$ , or

$$\vartheta_{On} = \vartheta_n = 4\lambda^{1/2}n + 3\lambda^{1/2} = 2\lambda^{1/2}[(2n + 1) + 1/2].$$

In the neighborhood of the points  $\vartheta_n$  we have ( $W = \vartheta_n + \Delta$ ,  $\Delta = E - \vartheta_n + i\varepsilon$ ,  $\alpha = -1/2 - n - \tilde{\Delta}$ ,  $\tilde{\Delta} = \Delta/4\sqrt{\lambda}$ )

$$\begin{aligned} \text{Im } \Omega_{\pi/2}(E + i0) &= -\frac{2\lambda^{1/4}}{\kappa_0^2\Gamma(-1/2 - n)} \text{Im } \Gamma(-n - \tilde{\Delta})|_{\varepsilon \rightarrow +0} = \\ &= \pi Q_{\pi/2|n}^2 \delta(E - \vartheta_n), \quad Q_{\pi/2|n} = \frac{2}{\kappa_0} \left[ \frac{\lambda^{3/4}(2n + 1)!!}{\sqrt{\pi}(2n)!!} \right]^{1/2}. \end{aligned}$$

Finally, we find

$$\sigma'_{\pi/2}(E) = \sum_{n=0}^{\infty} Q_{\pi/2|n}^2 \delta(E - \vartheta_n), \quad \text{spec } \hat{h}_{\pi/2} = \{\vartheta_n, n \in \mathbb{Z}_+\}.$$

A complete orthonormalized system of (generalized) eigenfunctions of  $\hat{h}_{\pi/2}$  is  $\{U_{\pi/2|n}(u) = Q_{\zeta|n}k_0O_{+1}(u; \vartheta_n), n \in \mathbb{Z}_+\}$ .

We obtain the same results for the case  $\zeta = -\pi/2$ .

b)  $\zeta = 0$

$$\Omega_0(E + i\varepsilon) = \frac{1}{\kappa_0\tilde{\gamma}(E + i\varepsilon)} = \frac{\Gamma(1/4 - (E + i\varepsilon)/4\sqrt{\lambda})}{2\lambda^{1/4}\Gamma(3/4 - (E + i\varepsilon)/4\sqrt{\lambda})}.$$

The function  $\Omega_0(E)$  is real for  $E$  if  $|\Omega_0(E)| < \infty$ . Therefore,  $\text{Im } \Omega_0(E + i0)$  can be not equal to zero only in the point  $\Omega_0(E) = \pm\infty$ , i. e., in the points  $\alpha = -n$ ,  $w(E_{0|n}) = w_{0|n} = n + 1/4$ ,  $n \in \mathbb{Z}_+$ , or

$$E_{0|n} = 4\lambda^{1/2}n + \lambda^{1/2} = 2\lambda^{1/2}(2n + 1/2).$$

In the neighborhood of the points  $E_{0|n}$  we have

$$\begin{aligned} \text{Im } \Omega_{00}(E + i0) &= \frac{1}{2\lambda^{1/4}\Gamma(1/2 - n)} \text{Im } \Gamma(-n - \tilde{\Delta})|_{\varepsilon \rightarrow +0} = \\ &= \pi Q_{0|n}^2 \delta(E - E_{0|n}), \quad Q_{0|n} = \left[ \frac{2(2n - 1)!!\lambda^{1/4}}{\sqrt{\pi}(2n)!!} \right]^{1/2}. \end{aligned}$$

Finally, we find

$$\sigma'_0(E) = \sum_{n=0}^{\infty} Q_{0|n}^2 \delta(E - E_{0|n}), \quad \text{spec } \hat{h}_0 = \{E_{0|n}, n \in \mathbb{Z}_+\}.$$

A complete orthonormalized system of (generalized) eigenfunctions of  $\hat{h}_0$  is  $\{U_{0|n}(u) = Q_{0|n}O_{+2}(u; E_{0|n}), n \in \mathbb{Z}_+\}$ .

c) General case of  $|\zeta| < \pi/2$

In this case we have

$$\sigma'_\zeta(E) = \frac{1}{\pi\kappa_0 \cos^2 \zeta} \operatorname{Im} \frac{1}{\tilde{\gamma}(E + i0) + \tan \zeta}.$$

The function  $\tilde{\gamma}(E)$  is real for real  $E$ . Therefore,  $\sigma'_\zeta(E)$  can be not equal to zero only in the points

$$\tilde{\gamma}(E_{\zeta|n}) = -\tan \zeta. \quad (2.17)$$

For the derivative of spectral function  $\sigma'_\zeta(E)$  we find

$$\begin{aligned} \sigma'_\zeta(E) &= \sum_{n=0}^{\infty} Q_{\zeta|n}^2 \delta(E - E_{\zeta|n}), \quad Q_{\zeta|n} = \left[ -\frac{1}{\kappa_0 \tilde{\gamma}'(E_{\zeta|n}) \cos^2 \zeta} \right]^{1/2}, \\ \tilde{\gamma}'(E_{\zeta|n}) &< 0, \quad \partial_\zeta E_{\zeta|n} = -1/[\tilde{\gamma}'(E_{\zeta|n}) \cos^2 \zeta] > 0 \end{aligned}$$

Let us study eq. (2.17) in more details. The function  $\tilde{\gamma}(E)$  has the properties:  $\tilde{\gamma}(E) = \kappa_0^{-1}|E|^{1/2} + O(|E|^{-1/2}) \rightarrow \infty$  as  $E \rightarrow -\infty$ ;  $\tilde{\gamma}(\vartheta_n \pm 0) = \pm\infty$ ,  $n \in \mathbb{Z}_+$ ;  $\tilde{\gamma}(E_{0|n}) = 0$ ,  $n \in \mathbb{Z}_+$ ;  $E_{0|n} < \vartheta_n < E_{0|n+1} < \vartheta_{n+1}$ ,  $n \in \mathbb{Z}_+$ . Then we find: in each energy interval  $(\vartheta_{n-1}, \vartheta_n)$ ,  $n \in \mathbb{Z}_+$ , for fixed  $\zeta \in (-\pi/2, \pi/2)$ , exists one solution of eq.(2.17)  $E_{\zeta|n}$  monotonically increasing from  $\vartheta_{n-1}$  through  $E_{0|n}$  to  $\vartheta_n$  when  $\zeta$  runs from  $-\pi/2 + 0$  through 0 to  $\pi/2 - 0$  (we set  $\vartheta_{-1} = -\infty$ ). Note that the equalities  $\lim_{\zeta \rightarrow \pi/2} E_{\zeta|n} = \lim_{\zeta \rightarrow -\pi/2} E_{\zeta|n+1} = \vartheta_n$  hold which illustrate the equivalence of the extensions with  $\zeta = -\pi/2$  and  $\zeta = \pi/2$ .

A complete orthonormalized system of (generalized) eigenfunctions of  $\hat{h}_\zeta$  is  $\{U_{\zeta|n}(u) = Q_{\zeta|n}U_\zeta(u; E_{\zeta|n}), n \in \mathbb{Z}_+\}$ .

## 2.8.4 Spectrum, $\lambda = 0$

Guiding functional, spectral function

$$\begin{aligned} U_\zeta(u; W) &= \frac{\kappa_0 \sin(W^{1/2}u)}{W^{1/2}} \sin \zeta + \cos(W^{1/2}u) \cos \zeta, \\ \sigma'_\zeta(E) &= \frac{1}{\pi} \operatorname{Im} \Omega_\zeta(E + i0), \quad \Omega_\zeta(W) = \frac{1}{\kappa_0} \frac{\kappa_0 \cos \zeta + iW^{1/2} \sin \zeta}{\kappa_0 \sin \zeta - iW^{1/2} \cos \zeta}. \end{aligned}$$

a)  $\zeta = \pm\pi/2$

$$\begin{aligned} \Omega_{\pm\pi/2}(W) &= i\kappa_0^{-2}W^{1/2}, \quad \Omega_{\pm\pi/2}(E + i0) = \begin{cases} i\kappa_0^{-2}E^{1/2}, & E \geq 0 \\ -\kappa_0^{-2}|E|^{1/2}, & E < 0 \end{cases}, \\ \sigma'_{\pm\pi/2}(E) &= \begin{cases} \pi^{-1}\kappa_0^{-2}E^{1/2}, & E \geq 0 \\ 0, & E < 0 \end{cases}, \quad \operatorname{spec} \hat{h}_{\pm\pi/2} = [0, \infty) = \mathbb{R}_+. \end{aligned}$$

b)  $\zeta = 0$

$$\Omega_0(W) = iW^{-1/2}, \quad \Omega_0(E + i0) = \begin{cases} iE^{-1/2}, & E > 0 \\ |E|^{-1/2}, & E < 0 \end{cases},$$

$$\sigma'_0(E) = \begin{cases} E^{-1/2}/\pi, & E > 0 \\ 0, & E < 0 \end{cases}, \quad \text{spec} \hat{h}_0 = [0, \infty) = \mathbb{R}_+.$$

c) General case  $|\zeta| < \pi/2$

In this case, we have

$$\sigma'_\zeta(E) = \frac{1}{\pi \cos^2 \zeta} \text{Im} \frac{1}{\omega_0(E + i0)}, \quad \omega_0(W) = -iW^{1/2} + \kappa_0 \tan \zeta.$$

i)  $E \geq 0$

$$\sigma'_\zeta(E) = \rho_\zeta^2(E), \quad \rho_\zeta(E) = \left( \frac{1}{\pi \kappa_0^2 \sin^2 \zeta + E \cos^2 \zeta} \right)^{1/2}.$$

The spectrum of  $\hat{h}_\zeta$  is simple and continuous,  $\text{spec} \hat{h}_\zeta = \mathbb{R}_+$

ii)  $E = -\tau^2 < 0, \tau > 0$

In this case we have

$$\omega_0(E) = \tau + \kappa_0 \tan \zeta.$$

We find:

If  $\zeta \in [0, \pi/2)$ , then  $\sigma'_\zeta(E) = 0$  and there are no spectrum points.

If  $\zeta \in (-\pi/2, 0)$ , then  $\text{Im} \Omega_\zeta(E + i0)$  can be different from zero in the point  $E_{\zeta|-1} = -\kappa_0^2 \tan^2 \zeta$  only and

$$\sigma'_\zeta(E) = Q_{\zeta|-1}^2 \delta(E - E_{\zeta|-1}), \quad Q_{\zeta|-1} = \sqrt{\frac{2\kappa_0 \sin |\zeta|}{\cos^3 \zeta}}.$$

Finally, we obtain

$$\text{spec} \hat{h}_\zeta = \begin{cases} \mathbb{R}_+, & \zeta \in [0, \pi/2) \\ \mathbb{R}_+ \cup \{E_{\zeta|-1}\}, & \zeta \in (-\pi/2, 0) \end{cases}.$$

A complete orthonormalized system of (generalized) eigenfunctions of  $\hat{h}_\zeta$  is

$$\begin{cases} \{U_{\zeta|E}(u), E \geq 0\}, & \zeta \in [0, \pi/2) \\ \{U_{\zeta|E}(u), E \geq 0; U_\zeta(u)\}, & \zeta \in (-\pi/2, 0) \end{cases}$$

$$U_{\zeta|E}(u) = \rho_\zeta(E) U_\zeta(u; E), \quad U_\zeta(u) = Q_{\zeta|-1} U_\zeta(u; E_{\zeta|-1}).$$

### 2.8.5 Spectrum, $\lambda < 0$

First we write down the parameters of the theory in this case:

$$\begin{aligned} \varkappa &= e^{-i\pi/4} |\lambda|^{1/4}, \quad \varkappa^{-1} = e^{i\pi/4} |\lambda|^{-1/4}, \quad \varkappa^2 = -i|\lambda|^{1/2}, \\ \rho &= -i|\lambda|^{1/2} u^2 = e^{-i\pi/2} |\lambda|^{1/2} u^2, \quad \alpha = 1/4 - i\tilde{w}, \quad \tilde{w} \equiv \tilde{w}_O = W/4|\lambda|^{1/2}, \end{aligned}$$

$$\tilde{\alpha} = 1/4 + i\tilde{w} = \overline{\alpha_O(\tilde{w})} = 1/2 - \alpha, \quad \tilde{w}(E) = E/4|\lambda|^{1/2} = \overline{\tilde{w}(E)}.$$

$$\Omega_\zeta(W) = -\frac{1}{\kappa_0} \frac{\tilde{\gamma}(W) \sin \zeta - \cos \zeta}{\tilde{\gamma}(W) \cos \zeta + \sin \zeta},$$

$$\tilde{\gamma}(E) = \frac{4\pi|\lambda|^{1/4}}{\kappa_0|\Gamma(\alpha)|^2(e^{-\pi\tilde{w}} + ie^{\pi\tilde{w}})}$$

a)  $\zeta = \pm\pi/2$

$$\sigma'_{O\pm\pi/2}(E) = -\frac{1}{\pi\kappa_0} \operatorname{Im} \tilde{\gamma}(E + i0) = -\frac{1}{\pi\kappa_0} \operatorname{Im} \tilde{\gamma}(E) \equiv \rho_{\pm\pi/2}^2(E),$$

$$\rho_{O\pm\pi/2}(E) = \frac{2|\lambda|^{1/8}e^{\pi\tilde{w}/2}}{\kappa_0|\Gamma(\alpha)|(e^{2\pi\tilde{w}} + e^{-2\pi\tilde{w}})^{1/2}}, \quad \operatorname{spec} \hat{h}_{O\pm\pi/2} = \mathbb{R}.$$

b)  $\zeta = 0$

$$\sigma'_0(E) = \frac{1}{\pi\kappa_0} \operatorname{Im} \frac{1}{\tilde{\gamma}(E + i0)} = \frac{1}{\pi\kappa_0} \operatorname{Im} \frac{1}{\tilde{\gamma}(E)} \equiv \rho_0^2(E),$$

$$\rho_0(E) = \frac{1}{2\pi} e^{\pi\tilde{w}/2} |\Gamma(\alpha)| |\lambda|^{-1/8}, \quad \operatorname{spec} \hat{h}_0 = \mathbb{R}.$$

c) Arbitrary  $\zeta$

$$\sigma'_\zeta(E) = \frac{1}{\pi} \operatorname{Im} \Omega_\zeta(E + i0) = \frac{1}{\pi} \operatorname{Im} \Omega_\zeta(E) =$$

$$= \frac{4|\lambda|^{1/4}}{\kappa_0^2} \frac{e^{\pi\tilde{w}} |\Gamma(\alpha)|^2}{e^{2\pi\tilde{w}} |\Gamma(\alpha)|^4 \sin^2 \zeta + \left( e^{-\pi\tilde{w}} |\Gamma(\alpha)|^2 \sin \zeta + \frac{4\pi|\lambda|^{1/4}}{\kappa_0} \cos \zeta \right)^2} \equiv \rho_\zeta^2(E),$$

$$\operatorname{spec} \hat{h}_\zeta = \mathbb{R}.$$

A complete orthonormalized system of (generalized) eigenfunctions of  $\hat{h}_\zeta$  is

$$\{U_{\zeta|E}(u) = \rho_\zeta(E) U_\zeta(u; E), \quad E \in \mathbb{R}\}$$

### 2.8.6 S.a. extensions of $\hat{H}_{O_s}$

$$\operatorname{spec} \hat{H}_{O_{\zeta_s}} = \operatorname{spec} \hat{h}_{O_{\zeta_s}}.$$

The eigenfunctions  $U_{\hat{H}_{O_{\zeta_s}}}(u)$  of the complete set of eigenfunctions of the operator  $\hat{H}_{O_{\zeta_s}}$  are equal to

$$U_{H_{O_{\zeta_s}}}(u) = \frac{1}{\sqrt{2}} U_{h_{O_{\zeta_s}}}(|u|).$$

## 2.9 S.a. extensions of $\hat{H}_{Oa}$

Literal consideration gives

$$\text{spec} \hat{H}_{O\zeta_a} = \text{spec} \hat{h}_{O\zeta_a},$$

the eigenfunctions  $U_{\hat{H}_{O\zeta_a}}(u)$  of the complete set of eigenfunctions of operator  $\hat{H}_{O\zeta_a}$  are equal to

$$U_{H_{O\zeta_a}}(u) = \frac{1}{\sqrt{2}} \varepsilon(u) U_{h_{O\zeta_a}}(|u|).$$

Note that the spectrum of a total s.a. Hamiltonian  $\hat{H}_{O\zeta_s\zeta_a} = \hat{H}_{O\zeta_s} \oplus \hat{H}_{O\zeta_a}$  is simple for  $\lambda > 0$ ,  $\zeta_s \neq \zeta_a$ , and twofold for  $\lambda > 0$ ,  $\zeta_s = \zeta_a$ , and for  $\lambda \leq 0$ .

## 2.10 Standard extension

If we consider a differential operation  $\check{H}_O$  (2.3) as acting on complete axis  $\mathbb{R}$ , a symmetrical operator  $\hat{H}_{O(\mathbb{R})}$  should be determine as follows:

$$\hat{H}_{O(\mathbb{R})} : \left\{ D_{H_{O(\mathbb{R})}} = \mathcal{D}(\mathbb{R}), \hat{H}_{O(\mathbb{R})}\psi(u) = \check{H}_O\psi(u), \forall \psi \in D_{H_{O(\mathbb{R})}} \right\}.$$

Adjoint operator  $\hat{H}_{O(\mathbb{R})}^+$  is

$$\hat{H}_{O(\mathbb{R})}^+ : \left\{ \begin{array}{l} D_{H_{O(\mathbb{R})}^+} = \{\psi_*, \psi'_* \text{ are a.c. on } \mathbb{R}, \psi_*, \check{H}_O\psi_* \in L^2(\mathbb{R})\} \\ \hat{H}_{O(\mathbb{R})}^+\psi_*(u) = \check{H}_O\psi_*(u), u \in \mathbb{R}, \forall \psi_* \in D_{H_{O(\mathbb{R})}^+} \end{array} \right\}.$$

Because  $\omega_{H_O^+}(\psi_*, \chi_*) = [\chi_*, \psi_*](u)|_{u \rightarrow \infty} - [\chi_*, \psi_*](u)|_{u \rightarrow -\infty} = 0$ , the operator  $\hat{H}_{O(\mathbb{R})}^+$  is symmetrical and, as consequence, s.a.. That means that there is only one s.a. extension of symmetrical operator  $\hat{H}_{O(\mathbb{R})}$ , the operator  $\hat{H}_{O(\mathbb{R})\epsilon} = \hat{H}_{O(\mathbb{R})}^+ = \hat{H}_{O(\mathbb{R})}$ ,  $D_{H_{O(\mathbb{R})\epsilon}} = D_{H_{O(\mathbb{R})}^+}$ . Because the inclusions

$$D_{H_{O(\mathbb{R})}^+} \supset \mathcal{D}(\mathbb{R}) \supset \mathcal{D}(\mathbb{R} \setminus \{0\})$$

are hold true and  $\hat{H}_{O(\mathbb{R})\epsilon} \mathcal{D}(\mathbb{R} \setminus \{0\}) = \check{H}_O \mathcal{D}(\mathbb{R} \setminus \{0\}) = \hat{H}_O \mathcal{D}(\mathbb{R} \setminus \{0\})$ , we obtain that  $\hat{H}_{O(\mathbb{R})\epsilon} \supset \hat{H}_O$ , i. e.,  $\hat{H}_{O(\mathbb{R})\epsilon}$  is some s.a. extension of symmetrical operator  $\hat{H}_O$ . This s.a. extension is specified by the bondary conditions  $\psi'_s(0) = 0$ ,  $\psi_a(0) = 0$ ,  $\psi_{s,a} \in \left( D_{H_{O(\mathbb{R})\epsilon}} \right)_{s,a}$ . thus, we find  $\hat{H}_{O(\mathbb{R})\epsilon} = \hat{H}_{O\zeta_s\zeta_a}$ ,  $\zeta_s = 0$ ,  $\zeta_a = \pm\pi/2$ .

## 3 One-dimensional Coulomb-like interaction

In this section we will consider the equation

$$\partial_x^2 \psi(x) + \left( \frac{3}{16x^2} - \frac{g}{|x|} + \mathcal{E} \right) \psi(x) = 0, \quad (3.1)$$

$$\mathcal{E} = |\mathcal{E}| e^{i\varphi_{\mathcal{E}}}, +0 \leq \varphi_{\mathcal{E}} \leq \pi, \text{Im } \mathcal{E} \geq 0,$$

where  $\hbar^2 \mathcal{E}/2m$  is complex energy,  $\hbar^2 g/2m$  is a coupling constant. This problem is a particular case of generalized Kratzer problem, which for has been solved in [7]. Here we will present those results, which are interesting for investigations of the spectra of dual theories. Going through the same steps as in Section 2, we find what follows.

### 3.1 Solution on the semiaxis $x > 0$

Introduce a new variable

$$z = 2Kx, \quad K = \sqrt{-\mathcal{E}} = \sqrt{|\mathcal{E}|} e^{i(\varphi_{\mathcal{E}} - \pi)/2} = \sqrt{|\mathcal{E}|} [\sin(\varphi_{\mathcal{E}}/2) - i \cos(\varphi_{\mathcal{E}}/2)] \quad \partial_x = 2K \partial_z, \quad \partial_x^2 = 4K^2 \partial_z^2,$$

and new function  $\phi(z) = z^{-1/4} e^{z/2} \psi(x)$ . Then we obtain

$$z \partial_z^2 \phi(z) + (1/2 - z) \partial_z \phi(z) - (1/4 + g/2K) \phi(z) = 0. \quad (3.2)$$

Eq. (3.2) is the equation for confluent hypergeometric functions, in the terms of which we can express solutions of eq. (3.1). We will use the following solutions:

$$\begin{aligned} C_{+1}(x; \mathcal{E}) &= \kappa_0^{-1/2} x^{3/4} e^{-z/2} \Phi(\alpha + 1/2, 3/2; z), \\ C_{+2}(x; \mathcal{E}) &= x^{1/4} e^{-z/2} \Phi(\alpha, 1/2; z), \\ C_{+3}(x; \mathcal{E}) &= \pi^{-1/2} \Gamma(\alpha + 1/2) x^{1/4} e^{-z/2} \Psi(\alpha, 1/2; z) = \\ &= C_{+2}(x; \mathcal{E}) - \frac{2\sqrt{2\kappa_0 K} \Gamma(\alpha + 1/2)}{\Gamma(\alpha)} C_{+1}(x; \mathcal{E}), \quad \alpha + 1/2 \neq -n, n \in \mathbb{Z}_+, \\ \alpha &\equiv \alpha_C = 1/4 + g/2K = 1/4 - w, \quad w \equiv w_C = -g/2K. \end{aligned}$$

#### 3.1.1 Asymptotics

Let  $x \rightarrow +0$

We have

$$\begin{aligned} C_{+1}(x; \mathcal{E}) &= \kappa_0^{-1/2} x^{3/4} + O(x^{7/4}), \quad C_{+2}(x; \mathcal{E}) = C_{+\text{as}2}(x) + O(x^{9/4}), \\ C_{+3}(x; \mathcal{E}) &= C_{+\text{as}2}(x) - \frac{2\sqrt{2K} \Gamma(\alpha + 1/2)}{\Gamma(\alpha)} x^{3/4} + O(x^{7/4}), \quad \alpha + 1/2 \neq -n, n \in \mathbb{Z}_+, \\ C_{+\text{as}2}(x) &= x^{1/4} + 2gx^{5/4}. \end{aligned}$$

Let  $x \rightarrow \infty$ ,  $\text{Im } \mathcal{E} > 0$  ( $\text{Re } K > 0$ ,  $-\pi/2 < \arg K \leq 0$ ,  $-\pi/2 < \arg z \leq 0$ )

$$\begin{aligned} C_{+1}(x; \mathcal{E}) &= \frac{\sqrt{\pi}(2K)^{\alpha-1}}{2\kappa_0^{1/2} \Gamma(\alpha + 1/2)} x^{-w} e^{z/2} (1 + O(x^{-1})) = O(x^{-\text{Re } w} e^{\text{Re } Kx}), \\ C_{+2}(x; \mathcal{E}) &= \frac{\sqrt{\pi}(2K)^{\alpha-1/2}}{\Gamma(\alpha)} x^{-w} e^{z/2} (1 + O(x^{-1})) = O(x^{-\text{Re } w} e^{\text{Re } Kx}), \\ C_{+3}(x; \mathcal{E}) &= \pi^{-1/2} \Gamma(\alpha + 1/2) (2K)^{-\alpha} x^w e^{-z/2} (1 + O(x^{-1})) = \\ &= O(x^{\text{Re } w_C} e^{-\text{Re } Kx}). \end{aligned}$$



### 3.1.2 The limit $\mathcal{E} = 0$

$$\begin{aligned}
C_{+1}(x; \mathcal{E}) &\rightarrow \kappa_0^{-1/2} x^{3/4} \Phi(g/2K, 3/2; 2Kx) = \frac{x^{1/4}}{2\sqrt{\kappa_0 g}} \sinh(2\sqrt{gx}), \\
C_{+1}(x; 0) &= \frac{x^{1/4}}{2\sqrt{\kappa_0 g}} \sinh(2\sqrt{gx}); \\
C_{+2}(x; \mathcal{E}) &\rightarrow x^{1/4} \Phi(g/2K, 1/2; 2Kx) = x^{1/4} \cosh(2\sqrt{gx}), \quad C_{+2}(x; 0) = x^{1/4} \cosh(2\sqrt{gx}); \\
C_{+3}(x; \mathcal{E}) &\rightarrow x^{1/4} \left[ \cosh(2\sqrt{gx}) - \frac{\sqrt{2K}\Gamma(\alpha_C + 1/2)}{\sqrt{g}\Gamma(\alpha_C)} \right]_{K \rightarrow 0} \sinh(2\sqrt{gx}) = x^{1/4} e^{-2\sqrt{gx}}, \\
C_{+3}(x; 0) &= x^{1/4} e^{-2\sqrt{gx}}
\end{aligned}$$

(we used a relation

$$\sqrt{2K}\Gamma(\alpha_C + 1/2)/\Gamma(\alpha_C) \Big|_{K \rightarrow 0} = \kappa\kappa_0^{-1/2}\Gamma(\alpha_O + 1/2)/\Gamma(\alpha_O) \Big|_{\kappa \rightarrow 0} = 2^{-1}\kappa_0^{-1/2}\sqrt{-W} = \sqrt{g}.$$

Thus obtained solutions are in agreement with direct solution of eq. (3.1) for  $\mathcal{E} = 0$ .

Note, that all solutions of eq.(3.1) are square-integrable at the origin and only the solution  $C_{+3}(x; \mathcal{E})$  is square-integrable at the infinity for  $\text{Im } \mathcal{E} > 0$ , i. e.,  $C_{+3}(x; \mathcal{E}) \in L^2(\mathbb{R}_+)$  for  $\text{Im } \mathcal{E} > 0$ .

It follows from the relation 9.212.1 of [4] that

$$\begin{aligned}
e^{-z/2}\Phi(\alpha, 1/2; z) &= e^{-Kx}\Phi(1/4 + g/2K, 1/2; 2Kx) = \\
&= e^{Kx}\Phi(1/4 - g/2K, 1/2; -2Kx), \\
e^{-z/2}\Phi(\alpha + 1/2, 3/2; z) &= e^{-Kx}\Phi(3/4 + g/2K, 3/2; 2Kx) = \\
&= e^{Kx}\Phi(3/4 - g/2K, 3/2; -2Kx),
\end{aligned}$$

i. e., the functions  $C_{+1}$  and  $C_{+2}$  are even functions of  $K$  ( for fixed rest parameters and  $x$ ). That means that  $C_{+1}$  and  $C_{+2}$  are real-entire functions of  $\mathcal{E}$ .

The Wronskians of the solutions of eq.(3.1) are

$$\begin{aligned}
\text{Wr}(C_{+1}, C_{+2}) &= \text{Wr}(C_{+1}, C_{+3}) = -\kappa_0^{-1/2}/2, \\
\text{Wr}(C_{+2}, C_{+3}) &= -\sqrt{2K} \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha)}, \quad \text{Wr}(C_{+2}, C_{+3})|_{\mathcal{E}=0} = -\sqrt{g}.
\end{aligned}$$

### 3.1.3 Solution on the semiaxis $x < 0$

For  $x < 0$ , we will use the solutions  $C_{-k}(x; \mathcal{E})$ ,

$$C_{-k}(x; \mathcal{E}) = C_{+k}(|x|; \mathcal{E}), \quad k = 1, 2, 3, \quad x < 0.$$

## 3.2 Symmetrical operator $\hat{H}_C$

For given a differential operation  $\check{H}_C \equiv \check{H}$ ,

$$\check{H} = -\partial_x^2 + \frac{g}{|x|} - \frac{3}{16x^2},$$

we determine the following symmetrical operator  $\hat{H}_C \equiv \hat{H}$ ,

$$\hat{H} : \begin{cases} D_H = \mathcal{D}(\mathbb{R} \setminus \{0\}), \\ \hat{H}\psi(x) = \check{H}\psi(x), \forall \psi \in D_H, \end{cases}.$$

### 3.3 Adjoint operator $\hat{H}_C^+$

$$\hat{H}_C^+ \equiv \hat{H}^+ : \begin{cases} D_{H^+} = \{\psi_*, \psi'_* \text{ are a.c. in } \mathbb{R} \setminus \{0\}, \psi_*, \hat{H}_C^+ \psi_* \in L^2(\mathbb{R})\} \\ \hat{H}^+ \psi_*(x) = \check{H}\psi_*(x), x \in \mathbb{R} \setminus \{0\}, \forall \psi_* \in D_{H^+} \end{cases}.$$

#### 3.3.1 Asymptotics

I)  $|x| \rightarrow \infty$

Because  $V(x) = g/|x| - 3/16x^2 \rightarrow 0$  as  $|x| \rightarrow \infty$ , we have:  $\psi_*, \psi'_* \rightarrow 0, \forall \psi_* \in D_{H^+}$ ,  $[\psi_*, \chi_*](x) \rightarrow 0 \forall \psi_*, \chi_* \in D_{H^+}$  as  $x \rightarrow \pm\infty$ .

II)  $x \rightarrow +0$

Because  $\check{H}\psi_* \in L^2(\mathbb{R})$ , we have

$$\check{H}\psi_*(u) = (-\partial_x^2 + g/|x| - 3/16x^2)\psi_*(x) = \eta(x), \eta \in L^2(\mathbb{R}).$$

General solution of this equation can be represented in the form

$$\begin{aligned} \psi_*(x) &= a_{+1}C_{+1}(x; 0) + a_{+2}C_{+2}(x; 0) + I(x), \\ \psi'_*(x) &= a_{+1}C'_{+1}(x; 0) + a_{+2}C'_{+2}(x; 0) + I'(x), \end{aligned}$$

where

$$\begin{aligned} I(x) &= 2\sqrt{\kappa_0} \left[ C_{+2}(x; 0) \int_0^x C_{+1}(y; 0)\eta(y)dy - C_{+1}(x; 0) \int_0^x C_{+2}(y; 0)\eta(y)dy \right], \\ I'(x) &= 2\sqrt{\kappa_0} \left[ C'_{+2}(x; 0) \int_0^x C_{+1}(y; 0)\eta(y)dy - C'_{+1}(x; 0) \int_0^x C_{+2}(y; 0)\eta(y)dy \right]. \end{aligned}$$

We obtain with the help of the Cauchy-Bunyakovskii inequality:

$$I(x) = O(x^{3/2}), I'(x) = O(x^{1/2}), x \rightarrow +0,$$

so that we have

$$\begin{aligned} \psi_*(x) &= a_{+1}\kappa_0^{-1/2}x^{3/4} + a_{+2}C_{+as2}(x) + O(x^{3/2}), \\ \psi'_*(x) &= (3/4)a_{+1}\kappa_0^{-1/2}x^{-1/4} + a_{+2}C'_{+as2}(x) + O(x^{1/2}). \end{aligned}$$

III)  $x \rightarrow -0$

Analogously, we obtain for  $x \rightarrow -0$ :

$$\begin{aligned} \psi_*(x) &= a_{-1}\kappa_0^{-1/2}|x|^{3/4} + a_{-2}C_{+as2}(|x|) + O(|x|^{3/2}), \\ \psi'_*(x) &= -(3/4)a_{-1}\kappa_0^{-1/2}|x|^{-1/4} - a_{-2}C'_{+as2}(|x|) + O(|x|^{1/2}). \end{aligned}$$

### 3.4 Sesquilinear form $\omega_{H^+}(\psi_*, \chi_*)$

$$\begin{aligned}
\omega_{H^+}(\chi_*, \psi_*) &= \omega_{+H^+}(\chi_*, \psi_*) + \omega_{-H^+}(\chi_*, \psi_*), \\
\omega_{+H^+}(\chi_*, \psi_*) &= \int_0^\infty \left[ \overline{\chi_*(x)} \check{H} \psi_*(x) - \overline{\check{H} \chi_*(x)} \psi_*(x) \right] dx = \\
&= - [\chi_*, \psi_*](x)|_{x \rightarrow +0} = \frac{1}{2\kappa_0^{1/2}} (\overline{a_{\chi_*+2}} a_{\psi_*+1} - \overline{a_{\chi_*+1}} a_{\psi_*+2}), \\
\omega_{-H^+}(\chi_*, \psi_*) &= \int_{-\infty}^0 \left[ \overline{\chi_*(x)} \check{H} \psi_*(x) - \overline{\check{H} \chi_*(x)} \psi_*(x) \right] dx = \\
&= [\chi_*, \psi_*](x)|_{x \rightarrow -0} = \frac{1}{2\kappa_0^{1/2}} (\overline{a_{\chi_*-2}} a_{\psi_*-1} - \overline{a_{\chi_*-1}} a_{\psi_*-2}),
\end{aligned}$$

such that we have

$$\begin{aligned}
\omega_{H^+}(\chi_*, \psi_*) &= \frac{1}{2\kappa_0^{1/2}} (\overline{\mathbf{a}_{\chi_*2}} \mathbf{a}_{\psi_*1} - \overline{\mathbf{a}_{\chi_*1}} \mathbf{a}_{\psi_*2}) = \frac{i}{4\kappa_0^{3/2}} (\overline{\mathbf{b}_{\chi_*}} \mathbf{b}_{\psi_*} - \overline{\mathbf{d}_{\chi_*}} \mathbf{d}_{\psi_*}), \\
\mathbf{a}_1 &= \begin{pmatrix} a_{+1} \\ a_{-1} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} a_{+2} \\ a_{-2} \end{pmatrix}, \\
\mathbf{b} &= \begin{pmatrix} b_+ \\ b_- \end{pmatrix} = \mathbf{a}_1 + i\kappa_0 \mathbf{a}_2, \quad \mathbf{d} = \begin{pmatrix} d_+ \\ d_- \end{pmatrix} = \mathbf{a}_1 - i\kappa_0 \mathbf{a}_2.
\end{aligned}$$

### 3.5 Self-adjoint hamiltonians

Because all s.a. hamiltonians,  $\hat{H}_{C\epsilon}$ , act on its domains as  $\check{H}_C$ , it should specify definition domains only. The definition domain  $D_{H_{C\epsilon}} \equiv D_{H_\epsilon}$  of s.a. operator  $\hat{H}_{C\epsilon} \equiv \hat{H}_\epsilon$  is determined by condition

$$\omega_{H^+}(\chi, \psi) = 0, \quad \forall \chi, \psi \in D_{H_\epsilon},$$

from which it follows

$$\mathbf{d}_\psi = U \mathbf{b}_\psi, \quad \forall \psi \in D_{H_{C\epsilon}}, \quad (3.3)$$

where  $U$  is an arbitrary, but fixed for given extension, unitary  $(2 \times 2)$ -matrix,  $U^+ U = 1$ . Thus, any s.a. hamiltonian is determined by assignment of unitary matrix  $U$  (we will denote the corresponding s.a. hamiltonian by  $\hat{H}_{CU} (\equiv \hat{H}_U$  in this section)),

$$\hat{H}_U^+ : \begin{cases} D_{H_U} = \{\psi : \psi \in D_{\check{H}}^*, \mathbf{d}_\psi = U \mathbf{b}_\psi\} \\ \hat{H}_U \psi(x) = \check{H} \psi(x), \quad x \in \mathbb{R} \setminus \{0\}, \quad \forall \psi \in D_U \end{cases}.$$

Thus, there exists a  $U(2)$ -family of s.a. extensions of the initial symmetric operator  $\hat{H}_C$ .

### 3.6 Parity conserving extensions

The introduction of the parity operator is the same as in Section 2. The  $U = U_P$  matrix also has the same properties. So we come to defining the elements of the matrix.

In the terms of the a.b. conditions, the obtained form of the matrix  $U_P$  means the following:

$$a_{s,a+1} \cos \zeta_{s,a} = \kappa_0 a_{s,a+2} \sin \zeta_{s,a}, \quad |\zeta_{s,a}| \leq \pi/2, \quad \zeta_{s,a} = -\pi/2 \sim \zeta_{s,a} = \pi/2, \quad (3.4)$$

or

$$\psi_{s,a}(x) = \begin{cases} a[\kappa_0^{1/2} x^{3/4} \sin \zeta_{s,a} + C_{+as2}(x) \cos \zeta_{s,a}] + O(x^{3/2}), & x > 0 \\ \pm a[\kappa_0^{1/2} |x|^{3/4} \sin \zeta_{s,a} + C_{+as2}(|x|) \cos \zeta_{s,a}] + O(x^{3/2}), & x < 0 \end{cases}, \quad x \rightarrow 0, \quad (3.5)$$

where  $\zeta_{s,a} = \varphi_{s,a}/2 - \pi/2$ . The inverse statement is true as well. Namely, if matrix  $U$  gives the boundary condition of the form (3.5) (or (3.4)) then that matrix  $U$  has the form (2.8) with  $\varphi_{s,a} = 2\zeta_{s,a} + \pi$ . In what follows, we change the notation of s.a. operator  $\hat{H}_{U_P}$  for  $\hat{H}_{\zeta_{s,a}} \equiv \hat{H}_{C\zeta_{s,a}}$ .

### 3.7 Extensions on semiaxis $\mathbb{R}_+$

#### 3.7.1 Differential operation $\check{h}_C$

$$\check{h}_C \equiv \check{h} = \check{H} = -\partial_x^2 + \frac{g}{|x|} - \frac{3}{16x^2}.$$

#### 3.7.2 Symmetrical operator $\hat{h}_C$

$$\hat{h}_C \equiv \hat{h} : \begin{cases} D_{h_C} \equiv D_h = \mathcal{D}(\mathbb{R}_+) \\ \hat{h}\psi(x) = \check{h}\psi(x), \quad \forall \psi \in D_h \end{cases}$$

#### 3.7.3 Adjoint operator $\hat{h}_C^+$

$$\hat{h}_C^+ \equiv \hat{h}^+ : \begin{cases} D_{h^+} = \{\psi_*, \psi'_* \text{ are a.c. in } \mathbb{R}_+, \psi_*, \check{H}\psi_* \in L^2(\mathbb{R}_+)\} \\ \hat{h}^+\psi_*(x) = \check{h}\psi_*(x), \quad \forall \psi_* \in D_{h^+} \end{cases}.$$

#### 3.7.4 Asymptotics of $\psi_* \in D_{h^+}$

Literally repeating the considerations of 3.3.1 we obtain:

- I)  $x \rightarrow \infty$   
 $[\psi_*, \chi_*](x) \rightarrow 0$  as  $x \rightarrow \infty, \forall \psi_*, \chi_* \in D_{h^+}$ .
- II)  $x \rightarrow 0$

$$\begin{aligned} \psi_*(x) &= a_{+1}\kappa_0^{-1/2}x^{3/4} + a_{+2}C_{+as2}(x) + O(x^{3/2}), \\ \psi'_*(x) &= (3/4)a_{+1}\kappa_0^{-1/2}x^{-1/4} + a_{+2}C'_{+as2}(x) + O(x^{1/2}). \end{aligned}$$

### 3.8 Sesquilinear form $\omega_{h^+}(\psi_*, \chi_*)$

$$\begin{aligned} \omega_{h^+}(\chi_*, \psi_*) &= \int_0^\infty \left[ \overline{\chi_*(x)} \check{h}\psi_*(x) - \overline{\check{h}\chi_*(x)} \psi_*(x) \right] dx = \\ &= - [\chi_*, \psi_*](x)|_{x \rightarrow 0} = \frac{1}{2\kappa_0^{1/2}} (\overline{a_{\chi_*+2}} a_{\psi_*+1} - \overline{a_{\chi_*+1}} a_{\psi_*+2}) = \\ &= \frac{i}{4\kappa_0^{3/2}} (\overline{b_{\chi_*}} b_{\psi_*} - \overline{d_{\chi_*}} d_{\psi_*}), \quad b = a_1 + i\kappa_0 a_2, \quad d = a_1 - i\kappa_0 a_2. \end{aligned}$$

### 3.9 Self-adjoint hamiltonians

Because all s.a. hamiltonians,  $\hat{h}_{C\epsilon} \equiv \hat{h}_\epsilon$ , act on their domains as  $\check{h}$ , we should specify only definition domains. The definition domain  $D_{h_\epsilon}$  of s.a. operator  $\hat{h}_\epsilon$  is determined by condition

$$\omega_{h^+}(\chi, \psi) = 0, \quad \forall \chi, \psi \in D_{h_\epsilon},$$

from which it follows

$$d_\psi = e^{i\varphi} b_\psi, \quad \forall \psi \in D_{h_{C\epsilon}}, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \sim 2\pi,$$

or, equivalently

$$a_1 \cos \zeta = \kappa_0 a_2 \sin \zeta, \quad \zeta = \varphi/2 - \pi/2, \quad -\pi/2 \sim \pi/2.$$

Thus, any s.a. hamiltonian is determined by assignment of unitary matrix  $U(1) = e^{i\varphi}$  (we will denote the corresponding s.a. hamiltonian by  $\hat{h}_{C\zeta} \equiv \hat{h}_\zeta$ ),

$$\hat{h}_\zeta : \begin{cases} D_{h_\zeta} \equiv \{\psi : \psi \in D_{\check{h}}^*, \quad a_1 \cos \zeta = \kappa_0 a_2 \sin \zeta\} \\ \hat{h}_\zeta \psi(x) = \check{h} \psi(x), \quad \forall \psi \in D_{h_\zeta} \end{cases}.$$

Equivalently, the boundary condition for  $\psi \in D_{h_\zeta}$  can be represented in the form

$$\psi(x) = a[\kappa_0^{1/2} x^{3/4} \sin \zeta + C_{+as2}(x) \cos \zeta] + O(x^{3/2}) \rightarrow 0. \quad (3.6)$$

Thus, there exists a  $U(1)$ -family of s.a. extensions  $\hat{h}_\zeta$  of the initial symmetric operator  $\hat{h}$ .

### 3.10 Self-adjoint extensions of $\hat{H}_s$

The Hilbert space  $L_s^2(\mathbb{R})$  is the space of all symmetric functions that are square integrable on  $\mathbb{R}$ . For these functions, the relations

$$(\chi, \psi) = 2(\chi, \psi)_+, \quad \omega_{H^+}(\chi, \psi) = 2\omega_{H^+}(\chi, \psi)_+ = 2\omega_{h^+}(\chi, \psi), \quad (3.7)$$

hold true, where

$$(\chi, \psi)_+ = \int_0^\infty \overline{\chi(x)} \psi(x) dx, \quad (3.8)$$

and  $\omega_{H^+}(\chi, \psi)_+ = \omega_{h^+}(\chi, \psi)$  is the sesquilinear form with respect to the scalar product (3.8).

Let us consider the isometry  $T: \psi \in \mathbb{R} \xrightarrow{T} \sqrt{2}\psi, \psi \in \mathbb{R}_+$ . Then

$$D_{H_s} \xrightarrow{T} D_h = \mathcal{D}(\mathbb{R}_+), \quad D_{H_s^+} = D_{\check{H}}^*(\mathbb{R}) \xrightarrow{T} D_{h^+} = D_{\check{h}}^*(\mathbb{R}_+). \quad (3.9)$$

It follows from eqs. (3.7) and (3.9) that there is one-to-one correspondence (the isometry  $T$ ) between s.a. extensions  $\hat{H}_{\zeta_s}$  of the symmetric operator  $\hat{H}_s$  in  $L_s^2(\mathbb{R})$  and s.a. extensions  $\hat{h}_\zeta$  of the symmetric operator  $\hat{h}$  in  $L^2(\mathbb{R}_+)$ :  $\hat{H}_{\zeta_s} \xleftrightarrow{T} \hat{h}_\zeta, \zeta_s = \zeta$ . Thus, the spectral analysis of s.a. operator  $\hat{H}_{\zeta_s}$  in  $L_s^2(\mathbb{R})$  is reduced to the spectral analysis of s.a. operator  $\hat{h}_\zeta, \zeta_s = \zeta$ , in  $L^2(\mathbb{R}_+)$ . Below, we represent this analysis.

### 3.10.1 Green function $G_{C\zeta}(x, y; \mathcal{E})$ , spectral function $\sigma_{C\zeta}(E)$

We find the Green function  $G_{C\zeta}(x, y; \mathcal{E}) \equiv G_\zeta(x, y; \mathcal{E})$  as the kernel of the integral representation

$$\psi(x) = \int_0^\infty G_\zeta(x, y; \mathcal{E}) \eta(y) dy, \quad \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_\zeta^+ - \mathcal{E})\psi(x) = \eta(x), \quad \text{Im } \mathcal{E} > 0, \quad (3.10)$$

for  $\psi \in D_{h_\zeta}$ , that is,  $\psi \in L^2(\mathbb{R}_+)$  and  $\psi(x)$  satisfies the boundary conditions (3.6). We find

$$G_\zeta(x, y; \mathcal{E}) = -\Omega_{C\zeta}(\mathcal{E})U_\zeta(x; \mathcal{E})U_\zeta(y; \mathcal{E}) - 2\kappa_0^{-1/2} \begin{cases} \tilde{U}_\zeta(x; \mathcal{E})U_\zeta(y; \mathcal{E}), & x > y \\ U_\zeta(x; \mathcal{E})\tilde{U}_\zeta(y; \mathcal{E}), & x < y \end{cases}, \quad (3.11)$$

$$\Omega_{C\zeta}(\mathcal{E}) \equiv \Omega_\zeta(\mathcal{E}) = \frac{2\tilde{\omega}_\zeta(\mathcal{E})}{\kappa_0^{1/2}\omega_\zeta(\mathcal{E})}, \quad \omega_{C\zeta}(\mathcal{E}) \equiv \omega_\zeta(\mathcal{E}) = \tilde{\gamma}(\mathcal{E}) \cos \zeta + \sin \zeta,$$

$$\tilde{\omega}_{C\zeta}(\mathcal{E}) \equiv \tilde{\omega}_\zeta(\mathcal{E}) = \tilde{\gamma}(\mathcal{E}) \sin \zeta - \cos \zeta, \quad \tilde{\gamma}(\mathcal{E}) = 2\sqrt{\frac{2K}{\kappa_0}}\gamma(\alpha), \quad \gamma(\alpha) = \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha)},$$

where we used relations

$$\begin{aligned} U_{C\zeta}(x; \mathcal{E}) &\equiv U_\zeta(x; \mathcal{E}) = \kappa_0 C_{+1}(x; \mathcal{E}) \sin \zeta + C_{+2}(x; \mathcal{E}) \cos \zeta, \\ \tilde{U}_{C\zeta}(x; \mathcal{E}) &\equiv \tilde{U}_\zeta(x; \mathcal{E}) = \kappa_0 C_{+1}(x; \mathcal{E}) \cos \zeta - C_{+2}(x; \mathcal{E}) \sin \zeta, \\ C_{+3}(x; W) &= U_\zeta(x; W)\tilde{\omega}_\zeta(W) - \tilde{U}_\zeta(x; W)\omega_\zeta(W). \end{aligned}$$

Note that  $U_\zeta(x; W)$  and  $\tilde{U}_\zeta(x; \mathcal{E})$  are real-entire solutions of eq. (3.1),  $U_\zeta(x; W)$  satisfies the boundary conditions (3.6), and the last term in the r.h.s. of eq. (3.11) is real for  $\mathcal{E} = E$  ( $\text{Im } \mathcal{E} = 0$ ).

### 3.10.2 Guiding functional

The guiding functional  $\Phi_{C\zeta}(\xi; W) \equiv \Phi_\zeta(\xi; W)$  is

$$\begin{aligned} \Phi_\zeta(\xi; \mathcal{E}) &= \int_0^\infty U_{C\zeta}(x; \mathcal{E}) \xi(x) dx, \quad \xi \in \mathbb{D}_\zeta = D_{h_{C\zeta}} \cap D_r(\mathbb{R}_+), \\ U_{C\zeta}(x; \mathcal{E}) &\equiv U_\zeta(x; \mathcal{E}) = \kappa_0 C_{+1}(x; \mathcal{E}) \sin \zeta + C_{+2}(x; \mathcal{E}) \cos \zeta. \end{aligned}$$

The guiding functional  $\Phi_\zeta(\xi; \mathcal{E})$  is simple and the spectrum of  $\hat{h}_{C\zeta}$  is simple. From the relation

$$U_\zeta^2(x_0; E) \sigma'_\zeta(E) = \frac{1}{\pi} \text{Im } G_\zeta(x_0 - 0, x_0 + 0; E + i0),$$

we find

$$\sigma'_\zeta(E) = -\frac{1}{\pi} \text{Im } \Omega_\zeta(E + i0).$$

### 3.10.3 Spectrum, $E \geq 0$

We have  $\varphi_{\mathcal{E}} = +0$ ,  $K = -i\sqrt{E}$ ,  $\alpha = 1/4 - i\tilde{w}$ ,  $\tilde{w} = -g/2\sqrt{E}$ ,

$$\begin{aligned}\tilde{\gamma}(E) &= 2\sqrt{\frac{2}{\kappa_0}}e^{-i\pi/4}E^{1/4}\frac{\Gamma(1 - (1/4 + i\tilde{w}))\Gamma(1/4 + i\tilde{w})}{|\Gamma(\alpha)|^2} = \\ &= \frac{4\sqrt{2}\pi E^{1/4}(e^{-\pi\tilde{w}C} - ie^{\pi\tilde{w}C})}{\kappa_0^{1/2}|\Gamma(\alpha_C)|^2(e^{2\pi\tilde{w}C} + e^{-2\pi\tilde{w}C})}.\end{aligned}$$

First we will study a question whether there exists the eigenvalue  $E = 0$ . Linearly independent solutions of eq. (3.1) for  $\mathcal{E} = 0$  are

$$C_{+1}(x; 0) = \frac{x^{1/4}}{2\sqrt{\kappa_0 g}} \sinh(2\sqrt{gx}), \quad C_{+3}(x; 0) = x^{1/4}e^{-2\sqrt{gx}}.$$

For  $g \leq 0$ , the square-integrable solutions are absent. For  $g > 0$ , there is one square-integrable solution  $C_{+3}(x; 0)$ . Representing the asymptotic of the function  $C_{+3}(x; 0)$  in the form

$$C_{+3}(x; 0) = (-2\kappa_0^{-1/2}g^{1/2})\kappa_0^{1/2}x^{3/4} + C_{2as}(x) + O(x^{7/4}), \quad x \rightarrow 0,$$

we find that this function is eigenfunction of s.a. hamiltonian  $\hat{h}_{\zeta_g}$ ,  $\zeta_g = \arctan(-2\kappa_0^{-1/2}g^{1/2})$ .

a)  $\zeta = \pm\pi/2$

$$\begin{aligned}-\frac{1}{\pi} \operatorname{Im} \Omega_{\pm\pi/2}(E) &= -\frac{1}{\pi} \operatorname{Im} \frac{2}{\kappa_0^{1/2}} \tilde{\gamma}(E) = \frac{8\sqrt{2}E^{1/4}e^{\pi\tilde{w}}}{\kappa_0|\Gamma(\alpha)|^2(e^{2\pi\tilde{w}} + e^{-2\pi\tilde{w}})} \equiv \\ &\equiv \rho_{C\pm\pi/2}^2(E) \equiv \rho_{\pm\pi/2}^2(E), \quad \lim_{E \rightarrow +0} \rho_{\pm\pi/2}^2(E) = \begin{cases} 0, & g \geq 0 \\ 4|g|^{1/2}/\pi\kappa_0, & g < 0 \end{cases}, \\ \sigma'_{\pm\pi/2}(E) &= \rho_{\pm\pi/2}^2(E), \quad \operatorname{spec} \hat{h}_{\pm\pi/2} = \mathbb{R}_+.\end{aligned}$$

b)  $\zeta = 0$

$$\begin{aligned}-\frac{1}{\pi} \operatorname{Im} \Omega_0(E) &= \frac{1}{\pi} \operatorname{Im} \frac{2}{\kappa_0^{1/2}\tilde{\gamma}(E)} = \frac{e^{\pi\tilde{w}C}|\Gamma(\alpha)|^2}{2\sqrt{2}\pi^2 E^{1/4}} \equiv \rho_{C0}^2(E) \equiv \rho_0^2(E), \\ \lim_{E \rightarrow +0} \rho_0^2(E) &= \begin{cases} 0, & g > 0 \\ 1/\pi|g|^{1/2}, & g < 0 \end{cases}, \\ \sigma'_0(E) &= \rho_0^2(E), \quad \operatorname{spec} \hat{h}_0 = \mathbb{R}_+.\end{aligned}$$

c)  $\zeta \neq 0, \pm\pi/2$ ,  $E > 0$

$$\begin{aligned}-\frac{1}{\pi} \operatorname{Im} \Omega_{\zeta}(E) &= \frac{2}{\pi\kappa_0^{1/2}} \operatorname{Im} \frac{e^{-\pi\tilde{w}} \cos \zeta - a \sin \zeta + ie^{\pi\tilde{w}} \cos \zeta}{e^{-\pi\tilde{w}} \sin \zeta + a \cos \zeta + ie^{\pi\tilde{w}} \sin \zeta} = \\ &= \frac{8\sqrt{2}E^{1/4}e^{\pi\tilde{w}}|\Gamma(\alpha)|^2}{(\kappa_0^{1/2}e^{-\pi\tilde{w}}|\Gamma(\alpha)|^2 \sin \zeta + 4\sqrt{2}\pi E^{1/4} \cos \zeta)^2 + \kappa_0 e^{2\pi\tilde{w}}|\Gamma(\alpha)|^4 \sin^2 \zeta} \equiv \\ &\equiv \rho_{C\zeta}^2(E) \equiv \rho_{\zeta}^2(E), \quad a = \frac{4\sqrt{2}\pi E^{1/4}}{\kappa_0^{1/2}|\Gamma(\alpha)|^2}.\end{aligned}$$

Now we can calculate  $\lim_{E \rightarrow +0} \rho_\zeta^2(E)$ .

i)  $g \leq 0$

$$\lim_{E \rightarrow +0} \rho_\zeta^2(E) = 4\pi^{-1}|g|^{1/2}(4|g| \cos^2 \zeta + \kappa_0 \sin^2 \zeta)^{-1}.$$

ii)  $g > 0, \zeta \neq \zeta_g$

$$\lim_{E \rightarrow +0} \rho_\zeta^2(E) = 0.$$

iii)  $g > 0, \zeta = \zeta_g$

$$\sigma'_{\zeta_g}(E) = 16g^{3/2}(1 + 4g/\kappa_0)\delta(E) + \rho_{\zeta_g}^2(E), \quad \rho_{\zeta_g}^2(0) = \lim_{E \rightarrow +0} \rho_{\zeta_g}^2(E) = 0,$$

$$\text{spec} \hat{h}_\zeta = \mathbb{R}_+.$$

Finally, we obtain for  $E \geq 0$ :

$$\sigma'_\zeta(E) = \begin{cases} \rho_\zeta^2(E), & \zeta \neq \zeta_g \\ 16g^{3/2}(1 + 4g/\kappa_0)\delta(E) + \rho_{\zeta_g}^2(E), & \zeta = \zeta_g \end{cases}.$$

Of course, the expressions of subsubsecs 3.10.3.a and 3.10.3.b are the limiting cases of the expressions of subsubsec 3.10.3.c.

### 3.10.4 Spectrum, $E < 0$ ,

In this case we have

$$\varphi_\mathcal{E} = \pi - 0, \quad K(E) = K|_{W=E} = \sqrt{|E|}, \quad \alpha = 1/4 - w, \quad w(E) = w|_{\mathcal{E}=E} = -g/2\sqrt{|E|},$$

$$\tilde{\gamma}(E) = 2\sqrt{2}\kappa_0^{-1/2}|E|^{1/4}\Gamma(\alpha + 1/2)/\Gamma(\alpha)$$

a)  $\zeta = \pi/2$

We find

$$-\Omega_{\pm\pi/2}(\mathcal{E}) = -\frac{2\tilde{\gamma}(\mathcal{E})}{\kappa_0^{1/2}} = -\frac{4\sqrt{2}|E|^{1/4}\Gamma(\alpha + 1/2)}{\kappa_0\Gamma(\alpha)}.$$

i)  $g \geq 0$

In this case,  $\Omega_{\pi/2}(E)$  is real and finite, such that  $\text{Im} \Omega_{\pi/2}(E) = \sigma'_{\pm\pi/2}(E) = 0$  and  $\text{spec} \hat{h}_{\pi/2} = \emptyset$ .

ii)  $g < 0, w(E) = |g|/2\sqrt{|E|}$ .

In this case, the function  $\Omega_{\pi/2}(E)$  is real for  $E$  when  $|\Omega_{\pi/2}(E)| < \infty$ . Therefore,  $\text{Im} \Omega_{\pi/2}(E + i0)$  can be not equal to zero only in the point  $\Omega_{\pi/2}(E) = \pm\infty$ , i. e., in the points  $\alpha = \alpha_{\pm\pi/2|n} = -1/2 - n$ ,  $w = w_{\pm\pi/2|n} = n + 3/4$ ,  $E = \vartheta_{Cn} \equiv \vartheta_n$ ,  $n \in \mathbb{Z}_+$ ,

$$\vartheta_n = -g^2[(2n + 1) + 1/2]^{-2}.$$

In the neighborhood of the points  $\vartheta_n$  we have ( $\mathcal{E} = \vartheta_n + \Delta$ ,  $\Delta = E - \vartheta_n + i\varepsilon$ ,  $\alpha(\mathcal{E}) = -1/2 - n - b\Delta$ ,  $b = |g||\vartheta_n|^{-3/2}/4$ )

$$-\text{Im} \Omega_{\pi/2}(E + i0) = -\frac{4\sqrt{2}|\vartheta_n|^{1/4}}{\kappa_0\Gamma(-1/2 - n)} \text{Im} \Gamma(-n - b\Delta)|_{\varepsilon \rightarrow +0} =$$

$$= \pi Q_{\pi/2|n}^2 \delta(E - \vartheta_n), \quad Q_{\pi/2|n} = \left[ \frac{8\sqrt{2}|\vartheta_n|^{7/4}(2n + 1)!!}{\sqrt{\pi}\kappa_0|g|2n!!} \right]^{1/2}.$$



Finally, we find

$$\sigma'_{\pi/2}(E) = \sum_{n=0}^{\infty} Q_{\pi/2|n}^2 \delta(E - \vartheta_n), \text{ spec } \hat{h}_{\pi/2} = \{\vartheta_n \mid n \in \mathbb{Z}_+\}.$$

We obtain the same results for the case  $\zeta = -\pi/2$ .

**b)  $\zeta = 0$**

In this case, we have

$$-\Omega_0(\mathcal{E}) = \frac{2}{\kappa_0^{1/2} \tilde{\gamma}(\mathcal{E})} = \frac{\Gamma(\alpha)}{\sqrt{2} K^{1/2} \Gamma(\alpha + 1/2)}, \quad -\Omega_0(E) = \frac{\Gamma(\alpha)}{\sqrt{2} |E|^{1/4} \Gamma(\alpha + 1/2)}.$$

i)  $g \geq 0$

In this case,  $\Omega_0(E)$  is real and finite, such that  $\text{Im } \Omega_0(E) = \sigma'_0(E) = 0$  and  $\text{spec } \hat{h}_0 = \emptyset$ .

ii)  $g < 0$ ,  $w(E) = |g|/2\sqrt{|E|}$ .

In this case, the function  $\Omega_0(E)$  is real for  $E$  when  $|\Omega_0(E)| < \infty$ . Therefore,  $\text{Im } \Omega_0(E + i0)$  can be not equal to zero only in the point  $\Omega_0(E) = \pm\infty$ , i.e., in the points  $\alpha = \alpha_{0|n} = -n$ ,  $w = w_{0|n} = n + 1/4$ ,  $E = E_{0|n}$ ,  $n = 0, 1, 2, \dots$ ,

$$|E_{0|n}|^{1/2} = |g|(2n + 1/2)^{-1}, \quad E_{0|n} = -g^2(2n + 1/2)^{-2}.$$

In the neighborhood of the points  $E_{0|n}$  we have ( $\mathcal{E} = E_{0|n} + \Delta$ ,  $\Delta = E - E_{0|n} + i\varepsilon$ ,  $\alpha(\mathcal{E}) = -n - b\Delta$ ,  $b = |g||E_{0|n}|^{-3/2}/4$ )

$$\begin{aligned} -\text{Im } \Omega_0(E + i0) &= \frac{1}{\sqrt{2} |E_{0|n}|^{1/4} \Gamma(1/2 - n)} \text{Im } \Gamma(-n - b\Delta)|_{\varepsilon \rightarrow +0} = \\ &= \pi Q_{0|n}^2 \delta(E - E_{0|n}), \quad Q_{0|n} = \left[ \frac{2\sqrt{2} |E_{0|n}|^{5/4} (2n - 1)!!}{\sqrt{\pi} |g| (2n)!!} \right]^{1/2}. \end{aligned}$$

Finally, we find

$$\sigma'_0(E) = \sum_{n=0}^{\infty} Q_{0|n}^2 \delta(E - E_{0|n}), \text{ spec } \hat{h}_0 = \{E_{0|n} \mid n \in \mathbb{Z}_+\}.$$

**c) General case  $|\zeta| < \pi/2$**

In this case, we have

$$\sigma'_\zeta(E) = \frac{2}{\pi \kappa_0^{1/2} \cos^2 \zeta} \text{Im } \frac{1}{\tilde{\gamma}(E + i0) + \tan \zeta}$$

The function  $\tilde{\gamma}(E)$  is real. Therefore,  $\sigma'_\zeta(E)$  can be not equal to zero only in the points, where

$$\tilde{\gamma}(E_{\zeta|n}) = -\tan \zeta, \tag{3.12}$$

such that we have

$$\begin{aligned} \sigma'_\zeta(E) &= \sum_n Q_{\zeta|n}^2 \delta(E - E_{\zeta|n}), \quad Q_{\zeta|n} = \left( -\frac{2}{\kappa_0^{1/2} \cos^2 \zeta} \frac{1}{\tilde{\gamma}'(E_{\zeta|n})} \right)^{1/2}, \\ \tilde{\gamma}'(E_{\zeta|n}) &< 0, \quad \partial_\zeta E_{\zeta|n} = -1/[\cos^2 \zeta \tilde{\gamma}'(E_{\zeta|n})] > 0. \end{aligned}$$

Let us study eq. (3.12) in more details.

i)  $g \geq 0$ ,

In this case, we have  $w(E) \leq 0$ ;  $\tilde{\gamma}(E) > 0$ ;  $\tilde{\gamma}(E) = 2^{3/2}\kappa_0^{-1/2}\Gamma^{-1}(1/4)\Gamma(3/4)|E|^{1/4} + O(|E|^{-1/4}) \rightarrow \infty$  as  $E \rightarrow -\infty$ . Eq. (3.12) has no solutions for  $\zeta \in (\zeta_0, \pi/2)$  and for any fixed  $\zeta \in (-\pi/2, \zeta_0)$  has one solution  $E_\zeta^{(-)} \in (-\infty, 0)$  monotonically increasing from  $-\infty$  to  $-0$  as  $\zeta$  run from  $-\pi/2 + 0$  to  $-0$  (let us remind that for  $g > 0$  and  $\zeta = \zeta_0$ , there exists the level  $E_{\zeta_0}^{(-)} = 0$ ).

ii)  $g < 0$ ,  $w(E) = |g|/2\sqrt{|E|}$

In this case, we have:  $\tilde{\gamma}(E) = 2^{3/2}\kappa_0^{-1/2}\Gamma^{-1}(1/4)\Gamma(3/4)|E|^{1/4} + O(|E|^{-1/4})$  as  $E \rightarrow -\infty$ ;  $\tilde{\gamma}(E_{0|n}) = 0$ ;  $\tilde{\gamma}(\vartheta_n \pm 0) = \pm\infty$ ;  $E_{0|n} < \vartheta_n < E_{0|n+1} < \vartheta_{n+1}$ . Then, in any domain  $(\vartheta_{n-1}, \vartheta_n)$ ,  $n \in \mathbb{Z}_+$ , for fixed  $\zeta \in (-\pi/2, \pi/2)$ , eq. (3.12) has one solution  $E_{\zeta|n}$  monotonically increasing from  $\vartheta_{n-1} + 0$  through  $E_{0|n}$  to  $\vartheta_n - 0$  as  $\zeta$  run from  $-\pi/2 + 0$  through  $0$  to  $\pi/2 - 0$  (we set  $\vartheta_{-1} = -\infty$ )

## 4 Comparison of the spectra of the theories of oscillator and Coulomb- like potential (1D Anyon)

Making the identifications

$$\begin{aligned} u &= \sqrt{x/\kappa_0}, \quad W = -4\kappa_0 g, \quad \lambda = -4\kappa_0^2 \mathcal{E}, \\ x &= \kappa_0 u^2, \quad \mathcal{E} = -\lambda/4\kappa_0^2, \quad g = -W/4\kappa_0, \Rightarrow \end{aligned} \quad (4.1)$$

we will get the following correspondence between oscillator and coulomb parameters and functions

$$\begin{aligned} K &= \sqrt{\lambda}/2\kappa_0 = \varkappa^2/2\kappa_0, \quad \sqrt{K} = \varkappa/\sqrt{2\kappa_0}, \quad z = \rho, \quad \sqrt{g} = \sqrt{-W}/2\sqrt{\kappa_0}, \\ \alpha_C &= \alpha_O, \quad w_C = w_O, \quad \tilde{w}_C = \tilde{w}_O \quad (E_C > 0, \quad \lambda < 0; \text{ see 8.3}), \\ \gamma_C(\alpha_C) &= \gamma_O(\alpha_O), \quad \omega_{C\zeta}(\mathcal{E}) = \omega_{O\zeta}(W), \quad \tilde{\omega}_{C\zeta}(\mathcal{E}) = \tilde{\omega}_{O\zeta}(W), \\ \Omega_{C\zeta}(\mathcal{E}) &= 2\kappa_0^{1/2}\Omega_{O\zeta}(W). \end{aligned}$$

Then we'll obtain

$$\begin{aligned} C_{+k}(x; \mathcal{E}) &= x^{1/4} O_{+k}(u; W), \quad k = 1, 2, 3, \\ C_{+as1}(x; \mathcal{E}) &= x^{1/4} O_{+as}(u; W), \end{aligned}$$

in agreement with eq.1.4.

It's easy to see, that for any fixed  $\zeta$ , to each point of continuous spectrum in the plane  $E_C, g$  corresponds a point of continuous spectrum in the plane  $E_O, \lambda$ , and to each point of discrete spectrum in the plane  $E_C, g$  corresponds a point of discrete spectrum in plane  $E_O, \lambda$ , while the image of the point, which is not a spectrum point in the plane  $E_C, g$ , is not point a spectrum point in the plane  $E_O, \lambda$ , and visa versa. Note, that a complete correspondence between the points of the spectra exists only if one takes into accounta “nonphysical”  $\lambda < 0$  in the case of oscillator.

The general statement on correspondence of the spectra of two problems is easily checked in the cases of  $\zeta = \pm\pi/2$  and  $\zeta = 0$ .

It is worth mentioning, that as was stated in previous sections, the complete orthonormalized system of (generalized) eigenfunctions of theories are  $U_{\zeta|E}(u) = \rho_{\zeta}(E)U_{\zeta}(u; E)$  for continuous spectrum and  $U_{\zeta|n}(u) = Q_{\zeta|n}U_{\zeta}(u; E)$  for discrete spectra. The connections between the normalized functions in two cases are

$$U_{C\zeta|E_C}(x;g) = \frac{\rho_{C\zeta}(E_C, g)}{\rho_{O\zeta}(E_O, \lambda)} x^{1/4} U_{O\zeta|E_O}(u; \lambda) = \sqrt{\frac{|u|}{2}} U_{O\zeta|E_O}(u; \lambda),$$

$$U_{C\zeta|n}(x;g) = \frac{Q_{C\zeta|n}(g)}{Q_{O\zeta|n}(\lambda)} x^{1/4} U_{O\zeta|n}(u; \lambda).$$

Note that the construction of the theory in the way described in this article automatically produces normalized wave functions. For the discrete spectrum for (in our terminology for standard extension  $\zeta_s = 0$ ,  $\zeta_a = \pm\pi/2$ ) we obtain the connection between the oscillator-anyon wave functions derived in [Ter-Ant] . [1].

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